1. Vector Calculus page 133 problem 32

Find any planes tangent to the curve \( z = x^2 - 6x + y^3 \) that are parallel to the plane \( 4x - 12y + z = 7 \).

The tangent plane to a curve \( f : \mathbb{R}^2 \to \mathbb{R} \) at a point \((a, b)\) is given by \( z = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b) \). For this function \( f_x = \frac{\partial f}{\partial x} = 2x - 6 \) and \( f_y = \frac{\partial f}{\partial y} = 3y^2 \). Thus the equation of the tangent plane to a curve at the point \((a, b)\) is \( z = (a^2 - 6a + b^3) + (2a-6)(x-a) + (3b^2)(y-b) \) or \((a^2 - 6a + b^3) = (6-2a)(x-a) - (3b^2)(y-b) + z \).

We want only tangent planes that are parallel to the plane \( 4x - 12y + z = 7 \). This implies that the tangent planes must have a normal vector that is a scalar multiple of \([4, -12, 1]\). Thus we arrive at the system of equations

\[
\begin{align*}
6 - 2a &= 4k \\
-3b^2 &= -12k \\
-1 &= -1k
\end{align*}
\]

The last equation forces \( k = 1 \). Thus we obtain the solution \( a = 1 \) and \( b = 2 \). This gives us the single tangent plane \(-4x + 12y - z = 13\).
2. If \( \lim_{x \to a} f(x) = M \) and \( \lim_{x \to a} g(x) = N \) then \( \lim_{x \to a} (f \circ g)(x) = MN \).

Proof:
Since \( \lim_{x \to a} f(x) = M \) there exists \( \delta_f \) such that \( |f(x) - M| < \frac{\epsilon}{3M} \) when \( \|x - a\| < \delta_f \). Also, \( \lim_{x \to a} g(x) = N \) implies there exists \( \delta_g \) such that \( |g(x) - N| < \frac{\epsilon}{3N} \) when \( \|x - a\| < \delta_g \). Let \( \delta_{fg} = \min(\delta_f, \delta_g) \).

Suppose that \( \|x - a\| < \delta_{fg} \). Consider \( |f(x)g(x) - MN| \). Because \( \|x - a\| < \delta_{fg} \leq \delta_f \), \( |f(x) - M| < \frac{\epsilon}{3M} \) or \( f(x) = M + \Delta_f \) with \( |\Delta_f| < \frac{\epsilon}{3M} \). Likewise, \( \|x - a\| < \delta_{fg} \leq \delta_g \), \( |g(x) - N| < \frac{\epsilon}{3N} \) or \( g(x) = N + \Delta_g \) with \( |\Delta_g| < \frac{\epsilon}{3N} \).

Thus,

\[
|f(x)g(x) - MN| = |(M + \Delta_f)(N + \Delta_g) - MN| \\
= |MN + \Delta_f M + \Delta_g N + \Delta_f \Delta_g - MN| \\
\leq |\Delta_f N| + |\Delta_g M| + |\Delta_f \Delta_g| \\
< |N| \frac{\epsilon}{3|N|} + |M| \frac{\epsilon}{3|M|} + \frac{\epsilon}{3} \\
= \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
= \epsilon.
\]

The last steps are true, because \( \frac{\epsilon^2}{9|M||N|} < \frac{\epsilon}{3} \) if \( \epsilon < 3|N||M| \). This is true if \( \epsilon \) is sufficiently small, and for a limit \( \epsilon \) becomes arbitrarily small.

Thus we have found a \( \delta_{fg} \) such that, \( |(f \circ g)(x) - MN| < \epsilon \) when \( \|x - a\| < \delta_{fg} \).

QED