COMPLETION OF A SEQUENCE OF CARLITZ

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Abstract. The exponential generating functions of $\{n^{n\pm m}\}_{n\geq 1}$ are given as rational functions of the tree function and then of the endofunction function, and coefficients of some of these rational functions are identified with certain generalized Stirling, Eulerian, and Bernoulli numbers.

In memory of Leonard Carlitz

1. Introduction: Carlitz' transform of formal power series.

The variable $\lambda = T(z) = \sum_{n\geq 1} n^{n-1}\frac{z^n}{n!}$ has been used extensively in the analysis of random mappings and random graphs [JKLP] [FO]. As formal power series, $z = \lambda e^{-\lambda}$, an identity known, in essence, to Euler. For integer values of m we define the sequence $G_m(\lambda) := R_m(z) := \sum_{n\geq 1} n^{n-m}\frac{z^n}{n!}$:

$$\dots$$

$$G_4(\lambda) = -\frac{1}{24}\lambda^4 + \frac{11}{36}\lambda^3 - \frac{7}{8}\lambda^2 + \lambda$$

$$G_3(\lambda) = \frac{1}{6}\lambda^3 - \frac{3}{4}\lambda^2 + \lambda$$

$$G_2(\lambda) = -\frac{1}{2}\lambda^2 + \lambda$$

$$G_1(\lambda) = \lambda$$

$$G_0(\lambda) = \frac{\lambda}{1-\lambda}$$

$$G_{-1}(\lambda) = \frac{\lambda}{(1-\lambda)^3}$$

$$G_{-2}(\lambda) = \frac{\lambda(1+2\lambda)}{(1-\lambda)^5}$$

$$G_{-3}(\lambda) = \frac{\lambda(1+8\lambda+6\lambda^2)}{(1-\lambda)^7}$$

$$\dots$$

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Carlitz [C] studied the numerators of the negatively subscripted functions. Since then, the notation $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$ for the "second-order Eulerian" numbers has been standardized in [GKP] (Table 270). Specifically we have the following transform for nonnegative m:

$$G_{-m}(\lambda) := \sum_{n\geq 1} n^{n+m}\frac{z^n}{n!} = \frac{\lambda}{(1-\lambda)^{2m+1}}\sum_{k=0}^{m}\left\langle\!\!\left\langle {m \atop k} \right\rangle\!\!\right\rangle\lambda^k$$

This is the main result in [C], where the integer coefficients were defined by their three-term recurrence relation. Combinatorial interpretations of $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$ were subsequently given by Riordan[R1] and Gessel and Stanley [GS]. We will give simple formulas for an integral operator taking G_i to G_{i+1} and its inverse operator. Thus we are able to explore the half of the sequence not studied by Carlitz, who used only the differential operator. The coefficients of the polynomials $G_m(\lambda)$, $m > 0$, have appeared previously in combinatorial settings (see [DB], [Quad]). The following theorem was proved by the first author in his solution of Problem 10638 in the American Mathematical Monthly (proposed by B. Conolly).

Theorem. $G_{m+1}(\lambda) = \int_0^\lambda G_m(\rho)\frac{1-\rho}{\rho}\, d\rho$.

Proof. The only step is to substitute $u = \rho e^{-\rho}$ at the right time:

$$G_{m+1}(\lambda) = R_{m+1}(z) = \int_0^z R_m(u)\frac{du}{u} = \int_0^\lambda R_m(\rho e^{-\rho})\frac{1-\rho}{\rho}\, d\rho = \int_0^\lambda G_m(\rho)\frac{1-\rho}{\rho}\, d\rho.$$

Corollary 1. $G_{m-1}(\lambda) = \frac{\lambda}{1-\lambda}G'_m(\lambda)$.

Proof. The proof is immediate. (cf. [C])

Corollary 2. *If $m > 0$, $G_m(\lambda)$ is a polynomial of degree m without constant term. If we write $G_m(\lambda) = \sum_{k=1}^m (-1)^{k-1} g(m,k)\lambda^k$, then $k\cdot g(m,k) = g(m-1,k) + g(m-1,k-1)$ for $m > 1$.*

Proof. Also immediate by definite integration. We note also that the coefficient of λ in $G_m(\lambda), m > 0$, is always 1.

2. Extended Remark on $g(m,k)$.

The tabulation of $\left\langle\!\!\left\langle {m \atop k} \right\rangle\!\!\right\rangle$ led Carlitz to ask for combinatorial interpretations. In contrast, the coefficients $g(m,k)$ are well known. In [DB], a definition of generalized Bernoulli numbers of negative degree is given from which we see that $g(m,k) = (-1)^k\binom{m-1}{k}B_{-m}^{(-k)}$.

Consider the recurrence relation

$$k\cdot g(m,k) = g(m-1,k) + g(m-1,k-1)$$

for $m, k \geq 1$ where we initialize by $g(m,k) = 0$ if $k > m$, and $g(m,1) = 1$. Since this is our definition, the verification of the following formulas is routine.(Compare with the appendices in [DB] and [Quad])

The unique solution may be expressed as a sum of "Egyptian Fractions":

$$g(m,k) = \frac{1}{k!} \sum_{1 \le i_1 \le i_2 \le \cdots \le i_{m-k} \le k} \frac{1}{i_1 i_2 \cdots i_{m-k}}$$

An alternative form is related to an iterated difference of $f(x) = \frac{1}{x^{m-k+1}}$:

$$g(m,k) = \frac{1}{k-1!} \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(-1)^j}{(j+1)^{m-k+1}}$$

From the recurrence for the $g(m,k)$'s also follow generating functions:

$$g(m,k) = [z^{m-k}] \prod_{p=1}^{k} \frac{1}{p-z}$$

which may be written in a "connection coefficient" formula:

$$\frac{1}{(x+1)(x+2)\cdots(x+k)} = \sum_{n=0}^{\infty} (-1)^n g(k+n,k) x^n$$

(linking $g(m,k)$ with the signless Stirling cycle numbers).

An integral of a polynomial over the hypercube yields the same numbers:

$$g(m,k) = \frac{1}{(k-1)!} \int_0^1 \int_0^1 \cdots \int_0^1 (1 - x_1 x_2 \cdots x_{m-k+1})^{k-1} dx_1 \; dx_2 \cdots dx_{m-k+1}$$

since it evaluates to the iterated difference form above after applying the binomial theorem.

The rational number $g(m,k)$ has the "natural denominator" $(k!)^{m-k+1}$. The corresponding numerators $N(m,k)$ form a triangle of integers generated by

$$N(m,k) = [z^{m-k}] \prod_{p=1}^{k} \frac{1}{1 - \frac{k!}{p} z}$$

and previously tabulated and studied in [DKB].

3. A Fractional Linear Transform.

If λ may be called the tree variable, then

$$\zeta = Z(z) = \sum_{n \ge 1} n^n \frac{z^n}{n!} = \frac{\lambda}{1-\lambda}$$

is the endofunction variable. When we express $R_m(z) = G_m(\lambda)$ in terms of ζ a beautiful phenomenon occurs: Carlitz' rational functions (m nonpositive) become polynomials in ζ with integer coefficients strongly related to numbers called "associated Stirling numbers of the second kind" by Riordan [R2] or "second-order Stirling numbers" (denoted $\{\{m,k\}\}$) by Fekete[F].

In the notation of [F], $G_{-m}(\lambda) = H_{-m}(\zeta) = (1+\zeta)^{m+1} \sum_{k=1}^{m} \{\{{m+k \atop k}\}\} \zeta^k$ (for m nonnegative). The fractional linear substitution $\lambda \to \frac{\zeta}{1+\zeta}$ thus produces relations between second-order Eulerian and second-order Stirling numbers (e.g. $\{\{{n+q \atop q}\}\} = \sum_{i=0}^{n} \binom{n-i-1}{q-i-1} \left\langle\!\left\langle {n \atop i} \right\rangle\!\right\rangle$).

The $\zeta-$expressions for positive m, if we write $H_m(\zeta) = (1+\zeta)^{-m}\sum_{k=1}^{m} h(m,k)\zeta^k$, contain a triangle of rational numbers $h(m,k)$ related to the numbers $g(m,k)$ in much the same way. Here we note that these coefficients satisfy the recurrence

$$k \cdot h(m,k) = h(m-1,k) + (m-k+1)\cdot h(m,k-1).$$

The sequence $H_m(\zeta) := R_m(z) = G_m(\lambda)$ from Section 1 becomes

$$\cdots$$

$$H_4(\zeta) = \frac{\zeta(1+\frac{17}{8}\zeta+\frac{14}{9}\zeta^2+\frac{7}{18}\zeta^3)}{(1+\zeta)^4}$$

$$H_3(\zeta) = \frac{\zeta(1+\frac{5}{4}\zeta+\frac{5}{12}\zeta^2)}{(1+\zeta)^3}$$

$$H_2(\zeta) = \frac{\zeta(1+\frac{1}{2}\zeta)}{(1+\zeta)^2}$$

$$H_1(\zeta) = \frac{\zeta}{1+\zeta}$$

$$H_0(\zeta) = \zeta$$

$$H_{-1}(\zeta) = \zeta(1+\zeta)^2$$

$$H_{-2}(\zeta) = \zeta(1+\zeta)^3(1+3\zeta)$$

$$H_{-3}(\zeta) = \zeta(1+\zeta)^4(1+10\zeta+15\zeta^2)$$

$$\cdots$$

4. Conclusion

The interplay between the Mobius transformation $\lambda \leftrightarrow \zeta$ and the combinatorial numbers in the rational expressions for $R_m(z)$ is quite pleasant. Since, perforce,

$$h(m,k) = \sum_{j=1}^{k}(-1)^{j-1}\binom{m-j}{k-j} g(m,j),$$

there is no lack of formulae for $h(m,k)$. Simple combinatorial interpretations for the $g(m,k)$'s are known (see, for example, [Quad]). Such an interpretation for $h(m,k)$ is desirable. The work of D. Loeb [L], reinterpreting $g(m,k)$ as a virtual Stirling number, may be useful in this. Some progress in this matter may be found in the E-print "Four Numerical Triangles in Nexus" at `http://saturn.math.uaa.alaska.edu/~smiley/numbertriangle.html`.

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