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Noncrossing Partitions Under Rotation and Reflection

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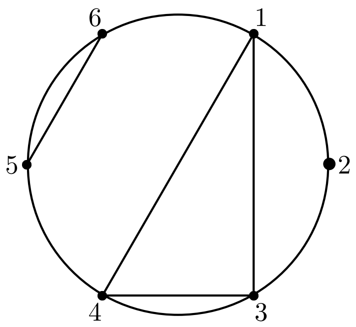
Abstract

We consider noncrossing partitions of $[n]$, taken up to rotation and/or reflection. Taken up to rotation, we exhibit a bijection to bicolored plane trees on n edges, and consider its implications. Taken up to reflection, we show that they are counted by the central binomial coefficients and that, somewhat surprisingly, the same count holds when they are taken under both rotation and reflection. The proof uses a pretty involution originating in work of Germain Kreweras. We conjecture that the "same count" phenomenon still holds when the word "noncrossing" is omitted.

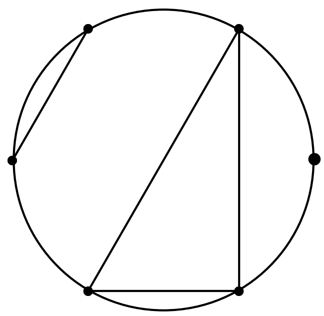
1 Introduction

The map $i \mapsto i+1 \pmod{n}$ on $[n]$ induces a map—the rotation operator R—on partitions π of $[n]$. Equivalence under repeated application of R divides them into rotation classes: $A(\pi) = \{R^i(\pi)\}_{i\geq 1}$. The *complement* of a partition π of $[n]$ is $C(\pi) := n+1-\pi$ (elementwise). It is easy to check that $C \circ R = R^{-1} \circ C$ and so the complement operation permutes rotation classes. We say a partition π is *self-complementary* if $C(\pi) = \pi$ and a rotation class A is self-complementary if $C(A) = A$. As we will see, a self-complementary rotation class need not contain any self-complementary partitions.

The operations rotation and complementation both preserve the noncrossing (NC) property of partitions. In particular, a rotation class consists entirely of NC partitions if it contains a single one. A NC rotation class may be represented by a polygon diagram with the labels removed, we'll call it a *plane polygon pattern* (PPP).



polygon diagram of the NC partition 134-2-56: labels fixed in place



polygon diagram of its rotation class is a PPP: no labels, rotate at will

Clearly, a NC partition is self-complementary if its labeled polygon diagram is invariant when flipped across a vertical line, and a NC rotation class is self-complementary if its plane polygon pattern is achiral, that is, invariant when flipped over (across any line).

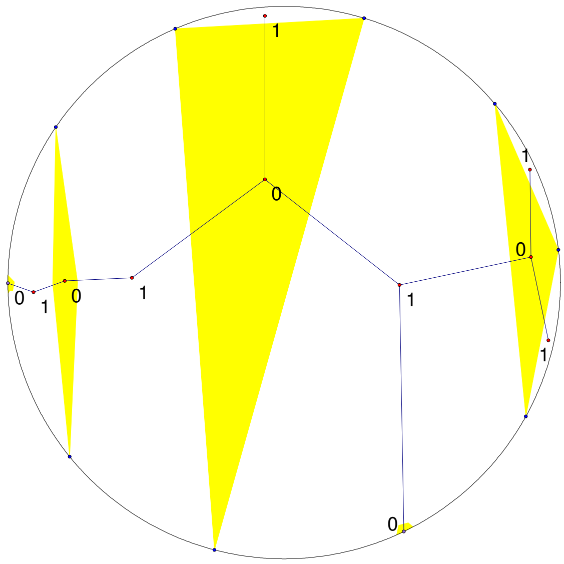
A *bicolored plane tree* is a plane tree (no root, no labels) in which each vertex is colored white or yellow (say) in such a way that adjacent vertices get different colors. The color of one vertex determines that of all the others and so the number of bicolored plane trees is at most twice the number of plane trees. In fact, it's a bit smaller.

2 Bijection

Direct enumeration of plane polygon patterns and bicolored plane trees is reviewed in the Appendix below. That they are equinumerous follows from the bijection between them illustrated on the following page. There are two immediate consequences of this bijection.

- The Catalan numbers (A000108) count bicolored plane trees with a distinguished edge. This is because the positioning of the labels on the circle can be captured by associating an edge in the tree with label 1, say the first tree edge encountered travelling clockwise from 1 around the polygon incident with 1. Thus the distinguished-edge bicolored plane trees on n edges are in correspondence with ordinary NC partitions of $[n]$, counted by C_n [8].

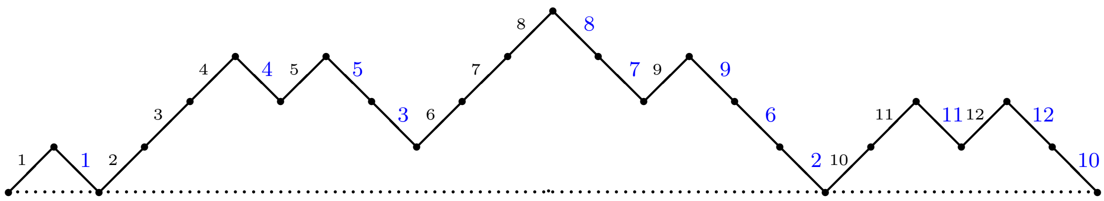
Figure 1: A bijective pair: $n = 10$ (points on circle; edges in tree). NC partition of type $(3, 3, 2, 1, 1)$. Leaves colored "0" are internal to polygons or singletons. Leaves colored "1" are between adjacent points in same part.



- On NC partitions, as well as on circular NC partitions, the statistics "# singletons" and "# adjacencies" have the same distribution, in fact a symmetric joint distribution. This is due to the correspondences yellow leaf $\leftrightarrow$ singleton block, and white leaf $\leftrightarrow$ adjacency, that is, two consecutive elements of $[n]$ in the same block. (Of course consecutive is taken here in the circular sense, so n and 1 are considered consecutive.) The symmetry of the joint distribution holds for unrestricted partitions too [3].

3 Counting self-complementary NC partitions

Next, we will show that the number of self-complementary NC partitions of $[n]$ is $\binom{n}{\lfloor n/2 \rfloor}$. To prepare for this count, recall a bijection from Dyck n-paths to NC partitions of $[n]$ that sends # peaks to # blocks. Given a Dyck n-path, number its upsteps left to right and then give each downstep the number of its matching upstep. The numbers on each descent (maximal sequence of contiguous downsteps) form the blocks of the corresponding NC partition.

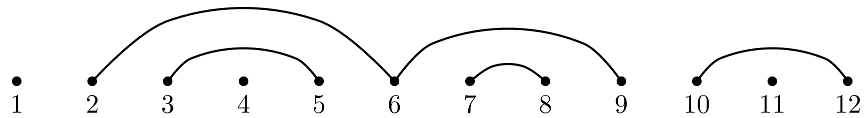


number downsteps,
each with the number of its matching upstep

Partition downstep labels by descents to get

$$1 - 4 - 5\,3 - 8\,7 - 9\,6\,2 - 11 - 12\,10,$$

a noncrossing partition with arc diagram



Under this bijection, peak downsteps correspond to largest block elements, and downsteps returning the path to ground level correspond to smallest elements in maximal blocks (a maximal block is one whose arcs would get wet if it rained, here there are 3 such: 1, 2 6 9, 10 12).

Theorem 1. *The number of self-complementary NC partitions of* $[n]$ *is* $|\mathcal{A}_n| = \binom{n}{\lfloor n/2 \rfloor}$.

Proof We give a bijective proof for n even. (The case n odd is similar and is omitted.) So suppose $n = 2m$. The right hand side clearly counts balanced paths of m upsteps and m downsteps. Now a NC partition π of $[2m]$ induces a partition τ of $[m]$ by intersecting its blocks with $[m]$. For τ_i a block of τ, set $\overline{\tau_i} = 2m + 1 - \tau$ (elementwise). If π is self-complementary then so is τ and each block of π has one of the three forms τ_i, $\overline{\tau_i}$ or $\tau_i \cup \overline{\tau_i}$ for some block τ_i of τ. The first two forms come in complementary pairs, the last form is permissible only if τ_i is a maximal block of τ (else π would have a crossing). So self-complementary NC partitions π of $[2m]$ correspond to NC partitions τ of $[m]$ in which each maximal block may (or not) be marked: a mark on τ_i indicating that $\tau_i \cup \overline{\tau_i}$ is a block in π, the absence of a mark indicating that τ_i, $\overline{\tau_i}$ are separate blocks of π. Using the NC partition $\leftrightarrow$ Dyck path correspondence above, these marked objects correspond in turn to Dyck m-paths with returns (to ground level) available for marking. Returns split a Dyck path into its components (Dyck subpaths whose only return is at the end). Flip over each component that terminates at a marked return to obtain a balanced m-path.

This is the desired bijection from self-complementary NC partitions of $[2m]$ to balanced m-paths. $\square$

4 Counting achiral plane polygon patterns

Theorem 2. *The set* $\mathcal{A}_n$ *of achiral plane polygon patterns (or self-complementary NC rotation classes) of* $[n]$ *is equinumerous with the set of self-complementary NC partitions of* $[n]$*, and hence* $|\mathcal{A}_n| = \binom{n}{\lfloor n/2 \rfloor}$.

To prove this, recall two related operations on NC partitions [4, 7] defined using polygon diagrams as illustrated in Figures 2,3,4 below.

In each case, new vertices (in blue) interleave the old vertices (in black) but their labelings differ. The new labels are then formed into maximal blocks subject only to: new polygons are disjoint from the old ones.

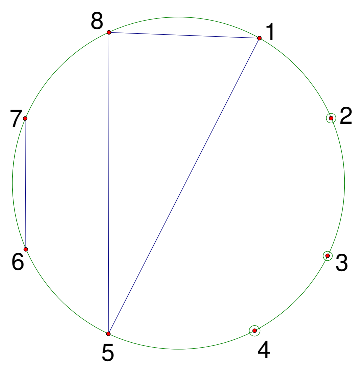


Figure 2: NC partition π by polygons

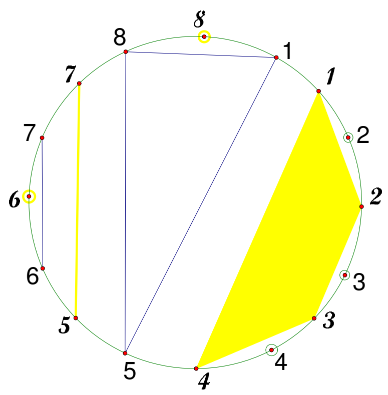


Figure 3: π with $H(\pi)$

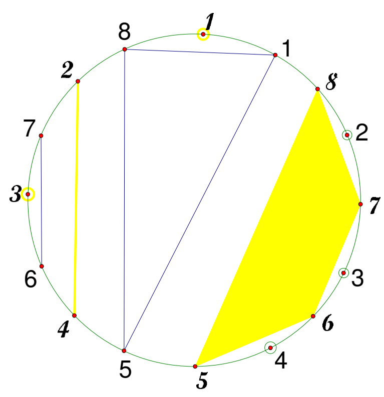


Figure 4: π with $T(\pi)$

It is clear from their defining diagrams that

$$H^2 = R^{-1}, \quad T^2 = I, \quad T = CH$$

and it is not hard to see that $TR = R^{-1}T$ and hence, by induction, $TR^i = R^{-i}T$ for all i. The following result is key to the bijection establishing the Theorem.

Proposition 1. *CT=TRC*

Proof $H^2 = R^{-1} \Rightarrow H = R^{-1}H^{-1} \Rightarrow CT = R^{-1}TC = TRC.$ $\square$

For $n \geq 3$, $H \neq R$ and we note in passing that the operations C, T, H, R on NC partitions of $[n]$ generate a dihedral group D_{2n} (of $4n$ elements) with presentation $\langle H, C : H^{2n} = C^2 = I,\ CHC^{-1} = H^{-1} \rangle$.

Next we define the notion of *complement order* on partitions in achiral NC rotation classes. Suppose $A \in \mathcal{A}_n$ and π is a partition in A. Then $C(\pi) \in A$ (because A is achiral) and so $C(\pi) = R^i(\pi)$ for some $i \geq 1$ ($i = n$ will do if $C(\pi) = \pi$). Define the complement order of π to be the minimal such $i \geq 1$.

Lemma 1.
(i) An achiral rotation class A contains at most 2 self-complementary partitions.
(ii) If $|A|$ is odd, then A contains exactly one self-complementary partition.
(iii) If $|A|$ is even, then either every partition in A has even complement order or every partition in A has odd complement order. In the former case, A contains 2 self-complementary partitions; in the latter case, none.

The proof is deferred. Theorem 2 will follow from this lemma if we can show that, among even-cardinality achiral NC rotation classes A in $\mathcal{A}_n$, there are just as many associated with even complement order as with odd. (Of course, $|A|$ even implies n even.) We claim the transpose T is a bijection, indeed an involution, that interchanges these two families. To see this, first suppose that $A \in \mathcal{A}_n$ has even cardinality, say $|A| = 2s$, and $\pi \in A$ has even complement order, say $C(\pi) = R^{2m}\pi$. Then, using Proposition 1,

$$C(T\pi) = TRC\pi = TR^{2m+1}\pi = R^{-(2m+1)}T\pi = R^{2s-2m-1}(T\pi)$$

and $T\pi$ has odd complement order. The other direction is similar, the desired bijection is established, and Theorem 2 follows.

Proof of Lemma 1 Suppose a rotation class A contains a self-complementary partition π. Then the complement of every other element of A is given by

$$CR^i\pi = R^{-i}C\pi = R^{-i}\pi \tag{1}$$

Now suppose $R^i\pi \in A$ is also self-complementary. It follows from (1) that $R^{2i}\pi = \pi$. Set $t = |A|$ so that $R^j\pi = \pi \Rightarrow t \mid j$. Hence $t \mid 2i$.

If t is odd, then $t \mid i$ and $R^i\pi = \pi$, implying that π is the only self-complementary partition in A. If t is even, say $t = 2s$, then $s \mid i$ and $R^i\pi$ is one of $R^s\pi$ and $R^{2s}\pi = \pi$. These facts establish part (i) and the "at most one" half of part (ii).

For the "at least one" half of part (ii), suppose $|A|$ is odd. Take $\pi \in A$. Since A is achiral, $C\pi = R^k\pi$ for some k and so the complement of each element of A is given by $CR^i\pi = R^{k-i}\pi$. If k is even, then $i = k/2$ makes $R^i\pi$ self-complementary. On the other hand, if k is odd, say $k = 2\ell + 1$ and $|A| = 2s + 1$, then $i = \ell - s$ makes $R^i\pi$ self-complementary. This establishes part (ii).

For part (iii), let $t := |A|$ be even. First, suppose some $\pi \in A$ has even complementary order k: $C(\pi) = R^k\pi$. Then $C(R^i\pi) = R^{k-i}\pi = R^{k-2i}R^i\pi$ and the powers of R that fix $R^i\pi$ are all $\equiv k - 2i \pmod{t}$ and hence even. Thus every element of A has even complementary order and $R^i\pi$ is self-complementary for $i = k/2$ and $i = (k + t)/2$. Similarly, if some element π of A has odd complementary order, then they all do, and the equation $C(R^i\pi) = R^i\pi$ has no solution. $\square$

5 Concluding Remarks

1. The transpose defined in Figure 4 above coincides with the restriction to NC partitions of the conjugate [3] defined on all partitions of $[n]$. In particular, the algorithmic definition of conjugate given in [3] provides a practical way to compute the transpose.

2. An analog of Theorem 2 appears to hold for arbitrary partitions: the number of self-complementary rotation classes of partitions on $[n]$ coincides with the number of self-complementary partitions of $[n]$. The proof of Lemma 1 goes through unchanged (it does not use the NC property). Unfortunately, the conjugate does not serve in the role of transpose to interchange the two relevant families in this larger setting, and it would be interesting to find an extension of the transpose that does.

3. The Appendix contains direct counts of the objects in the bijection of Section 2. The enumeration of bicolored plane trees has been significantly generalized in [1].

Appendix: Enumeration Formula for Unlabeled NC Partitions and Bi-colored Plane Trees (almost *ab initio*)

The partitions of the set $[n] = \{1, 2, \ldots, n\}$ (the decompositions of $[n]$ as a union of pairwise-disjoint, non-empty subsets) are counted by the sequence of Bell numbers. If the elements of $[n]$ are regarded as the set of labels of n otherwise indistinguishable objects, the unlabeled enumeration of partitions of these objects is the same as counting the partitions of the integer n.

If the labeled objects are situated around a circle with $1 \to 2 \to 3 \to \cdots \to n \to 1$ forming a simple cycle, we may define a *crossing* partition as any partition having two parts $\Pi_1 \supseteq \{m_1, m_3\}$ and $\Pi_2 \supseteq \{m_2, m_4\}$ with $m_1 < m_2 < m_3 < m_4$. The definition naturally transports to unlabeled points since it is equivalent to the existence of two chords which cross within the circle, such that the endpoints of one belong to one part, and the endpoints of the other to a different part.

Non-crossing partitions (partitions which are not crossing) are beautiful, and have been closely studied. Motzkin noted [5, last sentence]that the number of labelled non-crossing partitions, as a function of n, satisfies the Catalan recurrence, and in fact these are counted by the sequence of Catalan numbers.

In the unlabelled case, there are two candidate sequences: the leading one regards the circle as embedded in a plane with the points evenly spaced and counts non-crossing partitions inequivalent under rotations of the circle (we refer to these as *plane NC patterns*); the second identifies two partitions counted in the first which are the same after reflection across a diameter of the circle (we call these *chirally equivalent plane NC patterns*). Motzkin gave the beginning of the second sequence as $1, 2, 3, 6, 9, 24$. This contains an (almost certainly clerical) error: the value for $n = 5$ should be 10, not 9.

Since every NC partition can be depicted by drawing the convex hulls of each part, arriving at a diagram of convex k-gons, $k \in \{0, 1, \ldots n\}$, pairwise disjoint, with $[n]$ as the union of their vertex sets, the plane NC patterns may be thought of as these diagrams under rotational equivalence (the PPP's of Section 1). Two competing enumerative analogies are valid: if the circle is discarded entirely, then Bell:partitions of n::Catalan:plane NC patterns; if only the NC requirement is relaxed, then Bell:possibly crossing partition patterns::Catalan:plane NC patterns.

The essential fact needed in the direct enumeration of plane NC patterns was proven by V. Reiner [6]: if $n = kd$, $k \geq 2$, and the points are labelled $a_1, a_2, \ldots, a_n$, then the number of NC partitions having the property "a_i and a_j are in the same part if and only if $a_{i+d \pmod n}$ and $a_{j+d \pmod n}$ are in the same part" is $\binom{2d}{d}$. We refer to such NC partitions as *d-clickable* (suggested by analogy to clicking a physical dial with n positions through d positions and arriving at the same partition). Reiner gives two proofs; for the convenience of the reader we informally describe the bijection used in one of them. Using k distinct colors for the a's, relabel the points consecutively to make k monocolor intervals subscripted $1, \ldots, d$. Consider the 'unwrapped' doubly-infinite sequence

$$\ldots \mathbf{a}_{d-1}, \mathbf{a}_d, a_1, a_2, \ldots, a_d, A_1, \ldots, A_d, \mathsf{a}_1, \mathsf{a}_2 \ldots\ldots\ldots, \mathbf{a}_{d-1}, \mathbf{a}_d, a_1, \ldots$$

(here font/case is used to denote color). On the circle, find a part of the partition, say of size p, consisting entirely of a consecutive set of points. This will always be possible for an NC partition. In a set L place the subscript of the first (clockwise) element in the chosen part, and in a set R the subscript of the last. This will be possible if the part is proper. Remove all elements with subscripts equal to those in this part from the doubly-infinite sequence and from the circle. The resulting sequence still consists of equal length monocolor intervals in the k colors, and (after equispacing the remaining points on the circle) is $(d-p)$-clickable. Repeat the process until no such proper part remains to be chosen, at which time the sets L and R are equinumerous, but otherwise arbitrary, subsets of $[d]$. The number of ways of specifying such an L and R is easily seen to be $\binom{2d}{d}$.

To reverse the process, consider a copy of the original doubly-infinite sequence and for each element of L (resp. R) place a Left (resp. Right) parenthesis to the left (resp. right) of each symbol in the sequence with subscript equal to this element. When L and R are exhausted the partition may be decoded from the parenthesized string in the usual manner.

Figure 5 displays a partition of [24] which is 3-, 6-, and 12-clickable (we use partitioning walls instead of polygons for viewability). The generator (click) σ of $\mathbb{Z}_{24}$ may be thought of as a rotation of the diagram through $2\pi/24$ leaving the labels in place. The pictured partition is then a fixed point of σ^{3i}, $i = 0, \ldots, 7$. As one of the $\binom{6}{3}$ 3-clickables, it is a fixed point of all of these, including σ^3, σ^9, σ^{15}, and σ^{21}. As one of the superset of $\binom{12}{6}$ 6-clickables, it may not be invariant under those 4 rotations, but it must be a fixed point of σ^6, and σ^{18}, while as one of the $\binom{24}{12}$ 12-clickables, it need only be an invariant of $\{\sigma^{12}, \sigma^0\}$. These considerations generalize succinctly in the following enumeration.

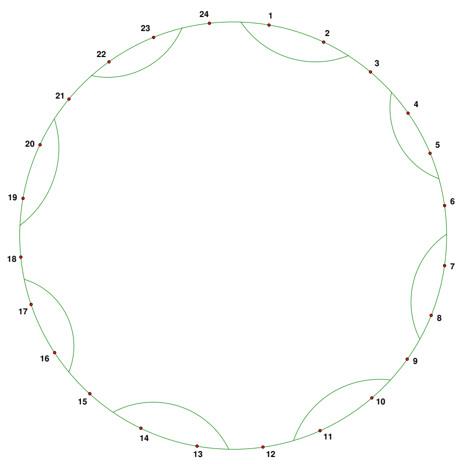


Figure 5: A 3-clickable pattern

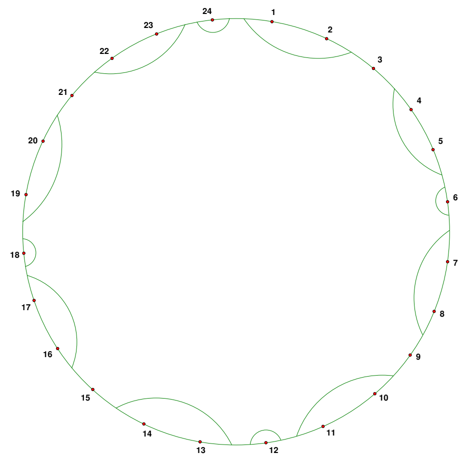


Figure 6: A 6-clickable pattern, not 3-clickable

Theorem 3. *The number of plane NC patterns of n points on a circle is*

$$\frac{1}{n}\left(\frac{1}{n+1}\binom{2n}{n}+\sum_{\substack{1\leq i<n\\ i|n}}\phi(\frac{n}{i})\binom{2i}{i}\right)$$

.

[Proof.]The Cauchy-Frobenius principle for the rotation group of order n counts our equivalence classes by summing over all group elements the number of objects (labelled NC partitions) invariant under the element, then dividing by the group order. The identity element accounts for the Catalan number as the left summand. It is easily verified that those non-identity group elements which fix all i-clickable partitions but not all j-clickable partitions for $j > i$ are exactly those of order $\frac{n}{i}$, and these number $\phi(\frac{n}{i})$. This gives the right summand, using the result of V. Reiner. QED

The number of (free) plane trees on n edges is known to be $\mathrm{FPT}(n) := \frac{1}{2n}\sum_{d|n}\phi(\frac{n}{d})\binom{2d}{d} - (C_n - C_{\frac{n-1}{2}})/2$ (A002995). Here and below C_m is understood to be 0 if m is not an integer, and the term involving Catalan numbers is an integer because C_m is odd iff the integer m has the form $2^k - 1$.

The *size* of a plane tree is its number of edges. The *subtrees* of a vertex are the plane trees obtained by deleting the vertex and its incident edges. A *center* of a plane tree is a vertex v that minimizes $\max\{\text{size}(T)\ :\ T \text{ a subtree of } v\}$. A plane tree either has a unique center or two adjacent centers. Deleting the connecting edge in the latter case leaves two ordered trees. Symmetry then implies that the number of bicolored plane tree on n edges is $2\,\text{FPT}(n) - C_{\frac{n-1}{2}}$ (A054357), since ordered trees are yet another manifestation of the Catalan numbers. Clearly this agrees with the Cauchy-Frobenius count above. M. Bousquet applied Cauchy-Frobenius ("Lemme de Burnside") to enumerate m-ary cacti in [2], applying a scheme due to Liskovets. His result includes bicolored plane trees as a special case.

n	NC Rotation Classes	NC Dihedral Classes	NC Chiral patterns $\div 2$
1	1	1	0
2	2	2	0
3	3	3	0
4	6	6	0
5	10	10	0
6	28	24	4
7	63	49	14
8	190	130	60
9	546	336	210
10	1708	980	728
11	5346	2904	2442
12	17428	9176	8252
13	57148	29432	27716
14	191280	97356	93924
15	646363	326399	319964
16	2210670	1111770	1098900
17	7626166	3825238	3800928
18	26538292	13293456	13244836
19	93013854	46553116	46460738
20	328215300	1642000028	164015272
21	1165060668	582706692	582353976
22	4158330416	2079517924	2078812492

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[8] Richard P. Stanley, *Enumerative Combinatorics Vol. 2*, Cambridge University Press, 1999. Exercise 6.19 and related material on Catalan numbers are available online at http://www-math.mit.edu/~rstan/ec/ .