Simple Proofs for Catalanian Forests and Paths

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1 Introduction

"Catalania" is a recent coinage [10, p. 256] related to the sequence of Catalan numbers
\[ C_n = \frac{1}{2n+1} \binom{2n}{n} \]. A visitor to Catalania may also encounter a three-parameter family of positive integers
\[ F(n, t, k) = \frac{k}{tn+k} \binom{tn+k}{n} \] (so that \( C_n = F(n, 2, 1) \)). The fun in this discovery is in describing \( F \) as a combinatorial, as well as arithmetical, generalization of \( C \).

Given integers \( t \geq 2, k \geq 1 \), consider the set \( \mathcal{P}(n, t, k) \) of lattice paths with steps \((1, 0)\) and \((0, 1)\) in the \((x, y)\) plane from \((0, 0)\) to \(((t-1)n+k-1, n)\) that never pass above the line \( L_t : (t-1)y = x \). It is well known that \( C_n \) enumerates these for \( t = 2, k = 1 \).

Similarly, we have the set \( \mathcal{A}(n, t, k) \) of plane (unlabelled) forests of exactly \( k \) components, each of which is a \( t \)-ary tree (all vertices have either 0 or \( t \) descendents), such that the total number of internal (non-leaf) vertices is \( n \).

Again \( C_n \) famously counts the case \( t = 2, k = 1 \).

In this note, we produce a bijection between paths \( \mathcal{P}(n, t, k) \) and forests \( \mathcal{A}(n, t, k) \), using an \( n \)-sequence of integers as intermediary, then give a quick verification that both families are counted by \( F(n, t, k) \).

2 Forests and \( b \)-sequences

A plane \( t \)-ary forest consists of an ordered set of rooted (ordered) trees whose vertices are either "internal" (having exactly \( t \) descendents), or are leaves (having 0 descendents). The vertices (\( N \) in number) are unlabelled, so a forest may be specified by the string \((d_1, \ldots, d_N)\), of 0’s and \( t \)-s, which results when the forest is traversed in preorder (root first, then descendent-trees; components left-to-right), and the number of descendents of each visited vertex is appended to the string. For example, '03000' has two components: an isolated vertex followed by a single root with its three children. The string '303000000' has three: a tree, the second child of whose root is internal, followed by two isolated vertices. Because a rooted tree has one more vertex than children (edges), it’s clear that \( N = tn + k \), where \( n \) is the number of internal vertices, and \( k \) is the
number of components, so that the leaf-count \((t - 1)n + k\). Not every string comes from a forest, however.

There is a simple necessary and sufficient condition:

**Proposition 1** An \(N\)-string of nonnegative integers comes from an ordered rooted forest with \(N\) vertices by the above procedure if and only if the sum of the last \(m\) integers in the string is less than \(m\), for \(1 \leq m \leq N\).

**Proof.** The proof is algorithmic and inductive. A forest is reconstructed beginning with the \(N\)th vertex, which must be a leaf, so \(d_N = 0\). As we step backward in the string, we maintain an ”orphan count”. If \(d_i = 0\), we place an isolated vertex to the left of the current forest and declare it an orphan, increasing the orphan count by 1. If \(d_i > 0\) we place a new vertex as the parent of the \(d_i\) most recently declared orphans, and declare this parent an orphan. This lowers the orphan count by \(d_i - 1\). In either case, the orphan count is incremented by \(d_i\). Any forest must have at least one parent-less node. The algorithm only fails by the placement of a vertex resulting in a nonpositive orphan count at the \(m\)th placement, or, equivalently, \(\sum_{j=1}^{m} (1 - d_{N-j+1}) \leq 0\).

QED

For our Catalan forests, we know that \(d_1 + \cdots + d_n = tn\) and \(N = nt + k\) and so the forest criterion is equivalent to \(d_1 + \cdots + d_k \geq \ell - k + 1\) for \(1 \leq \ell \leq N - 1\) and \(d_N = 0\). A string \(d_1d_2\ldots d_N\) is determined by the positions of the \(n\) \(t\)’s among the \((t - 1)n + k\) 0’s. Following Zaks [12], we denote by \(z_i\) the position of the \(i\)th \(t\), but immediately derive from it the reduced, still nonnegative, code \(b_i = z_i - i\). The key step in relating Catalan forests to lattice paths is the interpretation of the forest criterion on \(d\)’s in terms of the \(b\)’s.

**Proposition 2** A string \(d_1, \ldots, d_{nt + k}\) of \(n\) \(t\)’s and \((t - 1)n + k\) 0’s represents a \(t\)-ary forest with \(n\) internal vertices and \(k\) components if and only if, for the sequence \((b_1, \ldots, b_n)\) such that the \(i\)th \(t\) of the string is in position \(i + b_i\), we have \(0 \leq b_i \leq (i - 1)t - i + k\).

**Proof.** Assume that \(\ell\) is the smallest index for which the forest condition is violated, that is, \(d_1 + \cdots + d_{\ell} < \ell - k + 1\). Then \(d_{\ell}\) must be a 0 (because \(\ell\) is smallest), and there must be at least one \(t\) to the right of \(d_{\ell}\) in the string (since otherwise \(d_1 + \cdots + d_{\ell} = nt\)). If the first \(t\) to the right of \(d_{\ell}\) is the \(j\)th \(t\) in the string, then the forest condition must also be violated at the position immediately before this \(t\). This is position \(b_j + (j - 1)\). The violation reads

\[(j - 1)t = d_1 + \cdots + d_{b_j + j - 1} < (b_j + j - 1) - k + 1 = b_j - k + j.
\]

So \(b_j > (j - 1)t - j + k\) and the stated condition is equivalent to the impossibility of a violation. QED

### 3 \(b\)-sequences and Paths

Using the \(b\)-sequence encodings, we quickly obtain a bijection between \(A(n, t, k)\) and \(P(n, t, k)\).
This bijection is reflected geographically in Figure 1. If, for any path in \( \mathcal{P}(n,t,k) \), we view its vertical steps from the highest to the lowest, and record their distances to the line \( x = (t-1)n + k - 1 \), we obtain a sequence of \( n \) nonnegative integers satisfying exactly the conditions on the sequence of \( b \)'s derived above. This gives a bijection since all the \( b \)-sequences encode a path not crossing the slanted barrier line. We note that the parameter \( k \) is equal to \( (1 + \text{the horizontal distance from the target point of the path to the barrier}) \).

The classical counts of rooted plane forests \cite[Th. 5.3.10]{10} and/or lattice paths \cite{7} specialize to give \( F(n,t,k) = \frac{k^n}{n+k} \binom{n+t}{n} \). For completeness, we give a quick verification by an “old school” induction.

Fix \( t \geq 2 \) and denote \( |\mathcal{P}(n,t,k)| \) by \( g(n,k) \). Then

\[
g(n,k) = \sum_{i=t}^{i=t+k-1} g(n-1,i)
\]

because every path in \( \mathcal{P}(n,t,k) \) has its last vertical step from \( ((t-1)(n-1) - 1 + i, n-1) \to ((t-1)(n-1) - 1 + i, n) \) for some \( i = t, \ldots, t+k-1 \), and the summands count the legal paths from \((0,0)\) to \(((t-1)(n-1) - 1 + i, n-1)\). It is easy to check that \( g(1,k) = k \), so by induction we only need to calculate
The last equality is verified by algebraic manipulation. The summation step comes from the polynomial identity \([3, (1.48)]\)

\[
\sum_{i=t}^{i=t+k-1} \frac{i}{(n-1)t+i} \binom{(n-1)t+i}{n-1} = \sum_{j=0}^{j=k-1} \frac{j+t}{nt+j} \binom{nt+j}{n-1}
\]

\[
= \sum_{j=0}^{j=k-1} \frac{(nt+j) - t(nt-1)}{nt+j} \binom{nt+j}{n-1}
\]

\[
= \sum_{j=0}^{j=k-1} \binom{nt+j}{n-1} - t \binom{nt+j-1}{n-2}
\]

\[
= \binom{nt+k}{n} - \binom{nt}{n} - t \binom{nt+k-1}{n-1} + t \binom{nt-1}{n-1}
\]

\[
= \frac{k}{nt+k} \binom{nt+k}{n}
\]

4 Remarks

Invitation. I hope this economy tour of Catalania will entice the newcomer to further exploration. There could be no better guide than the online brochure of Stanley [9], introducing \(\approx 2^7\) points of interest, of which we have visited but three [namely his d)(trees), h)(paths), and s)(b-sequences)].

It is easy to check that all three of our models extend to the case \(t = 1\), so there is a balanced pyramid of Catalan Forest numbers \(F(n,t,k)\), \(1 \leq n,t,k\). Two projects are immediately available: a) extend other models in Stanley’s list to objects counted by \(F(n,2,k)\), \(F(n,t,1)\), or, optimally, \(F(n,t,k)\); b) extend the pyramid to more than three indicial dimensions. The second proposal is trivial for each model separately. The interest is in new bijections between models.

Rothe-Hagen proofs. It is amusing to note that an ancient identity \([3, (3.142)]\),
attributed to Rothe [8]

\[ \sum_{m=0}^{n} \frac{\ell}{mt+j} \binom{mt+j}{m} \frac{(n-m)t+\ell}{n-m} = \frac{j+\ell}{nt+j+\ell} \frac{(nt+j+\ell)}{n} \]

becomes almost obvious when interpreted using Catalanian forests. For other proofs, see [6, (1.2.6) Example 4], [1], [4].

An extension due to Hagen [5], [11], [13], reads, in our notation:

\[ \sum_{m=0}^{n} (p+qm) \cdot F(m, j, t) \cdot F(n-m, \ell, t) = \frac{p(j+\ell) + jqn}{j+\ell} F(n, j+\ell, t) \]

To get this, we only need to prove

\[ \sum_{m=0}^{n} m \cdot F(m, j, t) \cdot F(n-m, \ell, t) = \frac{jn}{j+\ell} F(n, j+\ell, t) \]

which is also easy using the forest model. To wit, the LHS counts \( t \)-ary forests with \( n \) internal nodes and \( j+\ell \) components with a distinguished internal node within the first \( j \) component trees (or none distinguished if the first \( k \) trees are isolated vertices). The RHS gives the fraction \( \frac{j}{j+\ell} \) of the \( t \)-ary forests with \( n \) internal nodes and \( j+\ell \) components and a distinguished internal node anywhere in the forest. But each of the latter objects may have its components cyclically shifted to produce \( j+\ell \) distinct ones (since the distinguished node moves!), exactly \( j \) of which satisfy the LHS condition.

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References


