3.5 Second Proof of the Implicit Function Theorem. There also exists a direct proof of this theorem that avoids the Inverse Function Theorem. Given $F_\alpha(x,y)$, a function $y = f(x)$ that satisfies $F[x,f(x)] = 0$ must also satisfy:

$$(1) \qquad \frac{\partial F_\alpha}{\partial x^i} + \sum_\beta \frac{\partial F_\alpha}{\partial y^\beta}\frac{\partial y^\beta}{\partial x^i} = 0$$

Hence we can find it, if it exists, by solving the differential equation:

$$(2) \qquad \frac{\partial y^\beta}{\partial x^i} = -\sum_\alpha \frac{\partial F_\alpha}{\partial x^i} \; G^\beta_\alpha$$

where the matrix $G^\beta_\alpha = (\partial F_\alpha / \partial y^\beta)^{-1}$, and hence exists in a neighborhood $U(x_o, y_o)$. So we need to check the integrability conditions of (2). Rather than doing so directly, we start from (1) and obtain:

$$\frac{\partial^2 F_\alpha}{\partial x^i \partial x^j} + \sum_\beta \frac{\partial^2 F_\alpha}{\partial x^i \partial y^\beta}\frac{\partial y^\beta}{\partial x^j} + \sum_\beta \frac{\partial^2 F_\alpha}{\partial y^\beta \partial x^j}\frac{\partial y^\beta}{\partial x^i} + \sum_{\beta,\gamma} \frac{\partial^2 F_\alpha}{\partial y^\beta \partial y^\gamma}\frac{\partial y^\beta}{\partial x^i}\frac{\partial y^\gamma}{\partial x^j}$$

$$+ \sum_\beta \frac{\partial F_\alpha}{\partial y^\beta}\frac{\partial^2 y^\beta}{\partial x^i \partial x^j} = 0$$

The first four terms taken as a whole are symmetric in i and j. Moreover $(\partial F_\alpha / \partial y^\beta)$ is a nonsingular matrix. Hence $\frac{\partial^2 y^\beta}{\partial x^i \partial x^j}$ is symmetric in i and j, and so the integrability conditions are satisfied.

We, therefore, choose $y^\beta = f^\beta(x)$ as solutions of (2) such that $y_o = f(x_o)$. These are unique. By our choice of x_o and y_o, $F[x_o, f(x_o)] = 0$. Moreover