3.5 Second Proof of the Implicit Function Theorem. There also exists a direct proof of this theorem that avoids the Inverse Function Theorem. Given \( F_\alpha(x,y) \), a function \( y = f(x) \) that satisfies \( F[x,f(x)] = 0 \) must also satisfy:

\[
(1) \quad \frac{\partial F_\alpha}{\partial x^i} + \sum_{\beta} \frac{\partial F_\alpha}{\partial y^\beta} \frac{\partial y^\beta}{\partial x^i} = 0
\]

Hence we can find it, if it exists, by solving the differential equation:

\[
(2) \quad \frac{\partial y^\beta}{\partial x^i} = -\sum_{\alpha} \frac{\partial F_\alpha}{\partial x^i} G^\beta_\alpha
\]

where the matrix \( G^\beta_\alpha = (\partial F_\alpha / \partial y^\beta)^{-1} \), and hence exists in a neighborhood \( U(x_0, y_0) \). So we need to check the integrability conditions of (2). Rather than doing so directly, we start from (1) and obtain:

\[
\begin{align*}
\frac{\partial^2 F_\alpha}{\partial x^i \partial x^j} + \sum_{\beta} \frac{\partial^2 F_\alpha}{\partial x^i \partial y^\beta} \frac{\partial y^\beta}{\partial x^j} + \sum_{\beta} \frac{\partial F_\alpha}{\partial y^\beta} \frac{\partial^2 y^\beta}{\partial x^i \partial x^j} + \sum_{\beta, \gamma} \frac{\partial F_\alpha}{\partial y^\beta} \frac{\partial y^\beta}{\partial x^i} \frac{\partial y^\gamma}{\partial x^j} + \sum_{\beta} \frac{\partial F_\alpha}{\partial y^\beta} \frac{\partial^2 y^\beta}{\partial x^i \partial x^j} \\
+ \sum_{\beta} \frac{\partial F_\alpha}{\partial y^\beta} \frac{\partial^2 y^\beta}{\partial x^i \partial x^j} = 0
\end{align*}
\]

The first four terms taken as a whole are symmetric in \( i \) and \( j \). Moreover \( (\partial F_\alpha / \partial y^\beta) \) is a nonsingular matrix. Hence \( \frac{\partial^2 y^\beta}{\partial x^i \partial x^j} \) is symmetric in \( i \) and \( j \), and so the integrability conditions are satisfied.

We, therefore, choose \( y^\beta = f^\beta(x) \) as solutions of (2) such that \( y_0 = f(x_0) \). These are unique. By our choice of \( x_0 \) and \( y_0 \), \( F[x_0, f(x_0)] = 0 \). Moreover