The Laguerre-Pólya Class
Non-linear operators and the Riemann Hypothesis

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The Riemann Hypothesis (1859)
The most famous of all unresolved problems.

\[ \xi \left( \frac{x}{2} \right) = 8 \int_0^\infty \Phi(t) \cos xt \, dt, \]

\[ \Phi(t) = \sum_{n=1}^{\infty} \left( \frac{2n^2}{\pi^2} e^{\frac{9}{2}t - \frac{3n^2}{\pi} e^{\frac{4}{t}}} \right) e^{-n^2 \pi e^{\frac{4}{t}}}. \]

"...es ist sehr wahrscheinlich, daß alle Wurzeln reell sind."

Theorem
The Riemann Hypothesis holds if and only if \( \xi \) belongs to the Laguerre-Pólya class.
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The Riemann Hypothesis holds if and only if \( \xi \) belongs to the Laguerre-Pólya class.
The Laguerre-Pólya Class (1914)

Functions in the Laguerre-Pólya class are uniform limits of polynomials all of whose zeros are real... and only the functions in the Laguerre-Pólya class enjoy this property.
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The Turán Inequalities (1948)
A necessary condition.

Theorem
If \( \varphi(x) = \sum_{k=0}^{\infty} \gamma_k k! x^k \) is a function in the Laguerre-Pólya class, then

\[
\gamma_{2k} - \gamma_k - 1 \gamma_k + 1 \geq 0
\]

for \( k = 1, 2, 3, \ldots \).
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A necessary condition.

Theorem

If \( \varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \) is a function in the Laguerre-Pólya class, then \( \gamma_k^2 - \gamma_{k-1} \gamma_{k+1} \geq 0 \) for \( k = 1, 2, 3, \ldots \).
The Riemann $\xi$ Function

$$\xi(x/2) = 8 \int_0^\infty \Phi(t) \cos xt \, dt,$$

$$\Phi(t) = \sum_{n=1}^\infty \left(2n^4 \pi^2 e^{9t} - 3n^2 \pi e^{5t}\right) e^{-n^2 \pi e^{4t}}.$$

Theorem (Csordas, Varga, Norfolk (1986))

The coefficients of the Riemann $\xi$ function satisfy the Turán inequalities.
Example

Construct a degree 5 polynomial $p(x)$ with zeros $x = -1$.

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$$p(x) = (x + 1)(x + 1)(x + 1)(x + 1)(x + 1)$$
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p(x) = (x + 1)(x + 1)(x + 1)(x + 1)(x + 1)
= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5
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\]

- Replace \( a_k \) with \( 3a_k^2 - 4a_{k-1}a_{k+1} + a_{k-2}a_{k+2} \).
Historical Background

Current Research

Non-linear operators.

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Construct a degree 5 polynomial $p(x)$ with zeros $x = -1$.

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\]

Replace $a_k$ with $3a_k^2 - 4a_{k-1}a_{k+1} + a_{k-2}a_{k+2}$.

$p(x)$ becomes $3 + 35x + 105x^2 + 105x^3 + 35x^4 + 3x^5$. 
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$p(x)$ becomes $3 + 35x + 105x^2 + 105x^3 + 35x^4 + 3x^5$.

...and the zeros remain real and negative.
Example (continued...)

\[ p(x) = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 \]
\[ = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^5 + a_5 x^5 . \]

The zeros of \( p(x) \) remain real and negative if \( a_k \) is replaced with:

- \( a_k^2 - a_{k-1} a_{k+1} \),
- \( 3a_k^2 - 4a_{k-1} a_{k+1} + a_{k-2} a_{k+2} \),
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\[ p(x) = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 \]
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The zeros of \( p(x) \) remain real and negative if \( a_k \) is replaced with:

- \( a_k^2 - a_{k-1} a_{k+1} \),
- \( p(x) \) becomes \( 1 + 15x + 50x^2 + 50x^3 + 15x^4 + x^5 \)
- \( 3a_k^2 - 4a_{k-1} a_{k+1} + a_{k-2} a_{k+2} \),
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\[ p(x) = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 \]
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- \( 3a_k^2 - 4a_{k-1}a_{k+1} + a_{k-2}a_{k+2} \),
- \[ p(x) \text{ becomes } 3 + 35x + 105x^2 + 105x^3 + 35x^4 + 3x^5 \]
- \( 10a_k^2 - 15a_{k-1}a_{k+1} + 6a_{k-2}a_{k+2} - a_{k-3}a_{k+3} \),

Lukasz Grabarek
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- \( 10a_k^2 - 15a_{k-1}a_{k+1} + 6a_{k-2}a_{k+2} - a_{k-3}a_{k+3} \),
  - \( p(x) \) becomes \( 10 + 100x + 280x^2 + 280x^3 + 100x^4 + 10x^5 \)
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\[ p(x) = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 \]
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The zeros of \( p(x) \) remain real and negative if \( a_k \) is replaced with:

- \( a_k^2 - a_{k-1} a_{k+1}, \)
- \( p(x) \) becomes \( 1 + 15x + 50x^2 + 50x^3 + 15x^4 + x^5 \)

- \( 3a_k^2 - 4a_{k-1} a_{k+1} + a_{k-2} a_{k+2}, \)
- \( p(x) \) becomes \( 3 + 35x + 105x^2 + 105x^3 + 35x^4 + 3x^5 \)

- \( 10a_k^2 - 15a_{k-1} a_{k+1} + 6a_{k-2} a_{k+2} - a_{k-3} a_{k+3}, \)
- \( p(x) \) becomes \( 10 + 100x + 280x^2 + 280x^3 + 100x^4 + 10x^5 \)

and infinitely many others....
The Main Result

Theorem (Grabarek (2010))

Let \( \varphi(x) = \sum_{k=0}^{\omega} a_k x^k \), \( 0 \leq \omega \leq \infty \), be a function in the Laguerre-Pólya class. If the zeros of \( \varphi(x) \) are real and negative, then the zeros remain real and negative after replacing \( a_k \) with

\[
\binom{2p - 1}{p} a_k^2 + \sum_{j=1}^{p} (-1)^j \binom{2p}{p - j} a_{k-j} a_{k+j} \quad (p = 1, 2, 3, \ldots)
\]