Minimum Sum Covers of Small Cyclic Groups

Mark A. Fitch* and Robert E. Jamison
Department of Mathematical Sciences
Clemson University
Clemson, SC 29634-0975
email: mfitch@clemson.edu
rejam@clemson.edu

Abstract

A subset S of an Abelian group $(A, +)$ is a sum cover provided every element $x \in A$ can be expressed in at least one way as a sum $x = s + t$ where s and t are in S. If $s \neq t$ is also required, then S is a strict sum cover. Sum covers form the natural finite analogue of the notion of an additive basis of the (positive) integers from additive number theory. In addition, they yield good constructions for the geometric problem of determining the minimum number of points that generate all slopes in a finite affine plane. There is no general construction technique for minimum cardinality sum covers. This paper reports on an exhaustive computer search yielding a census of all minimum cardinality sum covers in cyclic groups up to order 54.

1 Background Bounds

Our concern is with a natural extension of the notion of additive basis from additive number theory [4, 7] to finite cyclic groups of integers modulo n. Although the main results to be presented here are for cyclic groups only, most of the preliminaries extend immediately to general finite Abelian groups and, in fact, are more easily discussed in that context. Let S be a subset of a finite Abelian group $(A, +)$.

We define the set of sums from S to be $S + S := \{s + t : s, t \in S\}$, and the set of strict sums from S to be $S\# + \#S := \{s + t : s, t \in S, s \neq t\}$.

---

[*] Research supported in part by the German-American Fulbright Program

S is a *sum cover* of A iff $S+S=A$, and S is a *strict sum cover* of A iff $S\# + \#S = A$. Our main interest here is in the minimum cardinality $\mathrm{msc}(A)$ of a sum cover of A and the minimum cardinality $\mathrm{MSC}(A)$ of a strict sum cover of A. In addition, we will be concerned with the structure of the sets achieving these minima. We will use $\mathrm{msc}(n)$ and $\mathrm{MSC}(n)$ to denote $\mathrm{msc}(\mathbb{Z}_n)$ and $\mathrm{MSC}(\mathbb{Z}_n)$, respectively. Just by counting, it is easy to derive a fairly good lower bound, which unfortunately, is the only generally valid lower bound known.

(1.1) Theorem. Let $\mathrm{CLB}(n) := \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil$. Then for any Abelian group A of order n, $\mathrm{msc}(A) \geq \mathrm{CLB}(n) - 1$ and $\mathrm{MSC}(A) \geq \mathrm{CLB}(n)$.

Proof. Let S be a subset of A of cardinality $|S| = m$. Then S has at most $\binom{m}{2} + m$ sums and at most $\binom{m}{2}$ strict sums. Comparing these with n leads to necessary conditions on the size of m if S is a sum cover or strict sum cover:

$$\binom{m}{2} + m \geq n \text{ and } \binom{m}{2} \geq n.$$

These lead to the polynomial inequalities in m:

$$m^2 + m - 2n \geq 0 \text{ and } m^2 - m - 2n \geq 0.$$

In both cases, m must be integral and at least as large as the larger root. The larger roots differ by 1, so their ceilings do also. Thus the result follows. □

This yields a lower bound on the order of $\sqrt{2n}$. Using base b representation of the integers, it is possible to get an upper bound in the cyclic case on the order of $2\sqrt{n}$.

(1.2) Theorem. $\mathrm{MSC}(n) \leq 2\lceil \sqrt{n}\, \rceil$.

Proof. Let $b := \lceil \sqrt{n} \rceil$. Take $P := \{0, 1, 2, 3, \ldots, b-1\}$ and $Q := \{b, 2b, 3b, \ldots, (b-1)b\}$. Since $n \leq b^2$, the sums in $P+Q$, which are all strict, yield all non-zero integers from 1 to b^2-1. To get 0 as a strict sum, adjoin -1 to the set. Hence the set $S := P \cup Q \cup \{-1\}$ is a strict sum cover of $\mathbb{Z}_n$ of $2b$ elements. □

It is immediately obvious that this argument can be tweaked to yield a better bound. Unfortunately, the improvement is only slight and the verification becomes much more tedious. We have therefore elected to give only the simple, transparent case here to illustrate the general method. The improved bounds will be investigated more thoroughly in a forthcoming paper [1]. This construction does extend to general Abelian groups, but in a considerably weaker form. The construction does illustrate, however, a key theme of this paper and that is the role played by arithmetic progressions in sum covers. Indeed, the two sets P and Q above are both arithmetic progressions.

2 Slope Generating Sets

Sum covers can also be used in an interesting way to get a handle on the geometric problem of finding the smallest number of points in an affine plane required to generate lines in all possible directions.

(2.1) Theorem. Let F be a finite field of order q. Then there is a set of $\mathrm{MSC}(F,+)+1$ points in the affine plane $F \times F$ over F which pairwise determine lines having all possible slopes—including ∞.

Proof. Let S be a strict sum cover of the additive group $(F,+)$ of F. Let X be the set of points on the parabola $y = x^2$ of the form $X := \{(s, s^2) : s \in S\}$. Note that the slope joining (s, s^2) to (t, t^2) is $\frac{s^2-t^2}{s-t} = s+t$. Thus since X is a strict sum cover, X determines all possible finite slopes. To get the infinite slope, simply adjoin another point with the same x-coordinate as a point in X. $\Box$

Note that strictness is required since a sum $2s$ would correspond to a tangent line to the parabola through one of the points in X rather than a connecting (secant) line through two distinct points of X. Again, this argument may be tweaked to produce better bounds, as will be investigated in [1].

Also note that if F is not prime order, then its additive group is not cyclic and hence the above construction requires a sum cover over a noncyclic group. The above argument may be modified, using the ideas in [5], to take place on a hyperbola rather than a parabola. In this case, the relevant group is the multiplicative group of the field, which is cyclic of order $q-1$. And in this case, there are two "missing" slopes—namely, 0 and ∞. A more subtle argument using affinely regular polygons [2] allows a construction on an ellipse. This requires a sum cover for the cyclic group of order $q+1$ and leaves no "missing" slopes. Again the details are too involved to present in full here, and we have chosen to give the simplest case to give a hint of the methodology.

Notice that because of these constructions, $\mathrm{MSC}(n)$ is of special interest for the geometric slope generation problem when n is prime or differs by one from a prime power.

3 Holomorphic equivalence and the search for minimum sum covers

A holomorphy of an Abelian group $(A, +)$ is a mapping of the form $h(x) := \phi(x) + t$, where ϕ is an automorphism of A and t is any fixed element of A. As a composition of two bijections, any holomorphy is again a bijection. Applying an automorphism ϕ to two elements x and y will, by definition, apply the same automorphism to their sum: $\phi(x + y) = \phi(x) + \phi(y)$. Applying a translation $z \mapsto z + t$ to two elements x and y will apply twice that translation to their sum: $(x + t) + (y + t) = (x + y) + 2t$. Note that these transformations carry strict sums to strict sums. Thus if we apply a holomorphy to transform a set S into the set S^*, then a closely related holomorphy will carry the (strict) sum set of S onto the (strict) sum set of S^*. It follows that holomorphies preserve (strict) sum covers, and thus we shall regard two sum covers as *equivalent* iff there is a holomorphy that carries one onto the other.

As we have noted above, the minimum cardinality of a sum cover of $\mathbb{Z}_n$ is $O(\sqrt{n})$. However, determining the actual minimum and finding even one, let alone all, sum covers which achieve it can be a computationally daunting task. It is therefore not only convenient but necessary to find methods of pruning the search tree. Our approach involved three key steps:

1. We searched for sum covers only up to holomorphic equivalence. This allowed us, as described below, to assume that each cover started with 0 and 1.
2. We generated candidates in lexicographic order, so that the program produces the lexicographically first representative of each holomorphy class.
3. We conducted the search based on the length of the initial progression. For example, assume that we know that all covers with initial progression 0,1,2,3,4 have been generated and that we are now searching for those with a shorter progression of maximum length. Suppose we have some partial cover Q that we wish to attempt to complete to a full cover by adding elements. This represents an internal node in the search tree. If adding an element x to Q would produce an arithmetic progression holomorphically equivalent to 0,1,2,3,4, then we know that we have already found a representative of the holomorphy class of every cover that could result. Thus we may prune that branch from the search tree.

A consequence of this approach is that there is an enormous overhead in searching for all covers with the progression of greatest length, since no pruning occurs at this stage. However, once that stage is complete, pruning of the search tree does happen, and the computation speeds up. The more progression lengths are completed, the greater the pruning and hence the greater the speed. For this reason, we are interested in tabulating information on the longest progressions in sum covers.

Let us now show why we can assume that the sum cover contains 0 and 1. Notice that any (nonempty) set S in any Abelian group is equivalent via translation to a set containing 0, so 0 is no problem. Since the automorphisms of $\mathbb{Z}_n$ are all of the form $x \mapsto ax$ where a and n are relatively prime, over $\mathbb{Z}_n$ the holomorphies look like affine maps of the form $x \mapsto ax + t$ where $\gcd(a, n) = 1$.

(3.1) Theorem. In order for S in $\mathbb{Z}_n$ to be equivalent to a set containing 0 and 1, it is necessary and sufficient that S contain two elements whose difference is relatively prime to n.

Proof. Suppose $h : x \mapsto ax + t$ carries S to a set containing 0 and 1. Suppose u and v map to 0 and 1—that is, $au + t = 0$ and $av + t = 1$. Then $a(v - u) = (av + t) - (au + t) = 1 - 0 = 1$, so $v - u$ is invertible in $\mathbb{Z}_n$. Hence $v - u$ is relatively prime to n.

Conversely, if $v - u$ is prime to n, let a be the inverse of $v - u$ in $\mathbb{Z}_n$. Set $t = -au$ and let h be the holomorphy $h(x) = ax + t$. Then $h(u) = au + t = au - au = 0$ and $h(v) = av + t = av - au = a(v - u) = 1$. $\square$

A set in which the difference of any two elements is singular in $\mathbb{Z}_n$ (that is, shares a prime factor with n) is called a *singular set* in $\mathbb{Z}_n$. It is therefore important to know whether or not singular sum covers exist and, if so, for which n. If n has only one prime factor p, then in a singular set all elements would have to be congruent modulo p and hence could produce sums in only one congruence class modulo p. Hence there are no singular sum covers modulo a prime power. It is also rather easy to see, as first observed in [3] that there are no singular sum covers for n with exactly two prime factors. It is harder to show that the same is true for three prime factors [6] and much harder to extend this result to four prime factors [7].

At this point the argument breaks down. In [7], constructions are given for (strict) singular sum covers in certain $\mathbb{Z}_n$ where n is the product of five or more primes. In particular for $n = 2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$, there is a singular sum cover of $\mathbb{Z}_n$ of cardinality 190 consisting of all integers in $\mathbb{Z}_n$ which satisfy at least one of the following six congruences:

$$x \equiv 0 \pmod{30}, \quad x \equiv 28 \pmod{42}, \quad x \equiv 44 \pmod{66},$$

$$x \equiv 231 \pmod{462}, \quad x \equiv 55 \pmod{330}, \quad x \equiv 35 \pmod{210}.$$

This cover is minimal in that deleting any point destroys the covering property. However, it is not of minimum cardinality as the crude base b construction of Theorem (1.2) produces a sum cover for $\mathbb{Z}_{2310}$ with only 98 elements.

It is still not known for which n singular covers exist. In particular, it is not known whether or not there is a singular sum cover for the product of the first five odd primes. In any case, singular sum covers first appear well beyond the range tractable by exhaustive search.

(3.2) Theorem [7]. Every sum cover of $\mathbb{Z}_n$ for $n < 2310$ is holomorphically equivalent to one which contains 0 and 1.

4 Arithmetic progressions and deficits

An arithmetic progression of length k with difference d in an Abelian group $(A, +)$ is a sequence $x_1, x_2, \ldots, x_k$ such that $x_{i+1} - x_i = d$ for all i. The next two results discuss invariants of arithmetic progressions under the action of holomorphies.

(4.1) Theorem. A holomorphy takes an arithmetic progression of length k with difference d into an arithmetic progression of length k with difference d^* where d and d^* are in the same orbit of A under the action of the automorphism group $\mathrm{Aut}(A)$ of A.

Proof. Let $x_1, x_2, \ldots, x_k$ be an arithmetic progression with difference d. Let $y_i := \phi(x_i) + t$ where ϕ is an automorphism of A and t is a constant translation term. Then

$$\begin{aligned} y_{i+1} - y_i &= (\phi(x_{i+1}) + t) - (\phi(x_i) + t) \\ &= \phi(x_{i+1}) - \phi(x_i) \\ &= \phi(x_{i+1} - x_i) \\ &= \phi(d) \end{aligned}$$

Hence $d^* = \phi(d)$ is the constant difference for the y_i. $\square$

(4.2) Theorem. If A is a cyclic group, then two elements d and d^* are in the same orbit of A under the action of the automorphism group $\mathrm{Aut}(A)$ of A iff d and d^* have the same order.

Proof. $(\rightarrow)$ is clear since automorphisms preserve order.

$(\leftarrow)$ Let A be the group $\mathbb{Z}_n$ where $n = qm$. The subgroup $< q >$ generated by q is the unique subgroup of $\mathbb{Z}_n$ of order m. Its generators are precisely the elements of order m in $\mathbb{Z}_n$. Now any generator of $< q >$ has the form kq where $\gcd(k,m) = 1$. Hence to prove the result, it suffices to find an automorphism of $\mathbb{Z}_n$ that carries the principal generator q to kq. That is, we are seeking an invertible a in $\mathbb{Z}_n$ such that $aq = kq \bmod n$.

Write $q = rs$, where $s = \gcd(q,m)$, so that $\gcd(r,m) = 1$. Now since r and m are relatively prime, the congruences

$$x \equiv k \pmod{m} \text{ and } x \equiv 1 \pmod{r}$$

have a simultaneous solution, say a. We claim $\gcd(a,n) = 1$. Indeed $n = qm = rsm$, so any prime p that divides n must divide one of r, s, or m. If $p|r$, then p cannot divide a since $a \equiv 1 \ (r)$. If $p|s$, then $p|m$ since by choice of $s = \gcd(q,m)$, $s|m$. Hence $p|m$ is the only remaining case. But then p cannot divide a since $a \equiv k \ (m)$ and by assumption k is relatively prime to m.

Hence a is invertible and $x \mapsto ax$ defines an automorphism of $\mathbb{Z}_n$. Since $a \equiv k \ (m)$, we have $a = k + tm$ for some t. Hence applying the automorphism to q, we get $aq = (k+tm)q = kq + tmq = kq + tn = kq \ (n)$ as desired. $\square$

As noted in the last section, we are interested in the role played by arithmetic progressions in the structure of sum covers. The first result above shows that the length of the longest progression is a holomorphy invariant, and the second result gives an easy way to distinguish the holomorphy type of a progression based on the order of its constant difference. When n is prime, all (nonzero) elements are generators and hence have order n. Hence when n is prime, all progressions of the same length are holomorphically equivalent. When n is not prime, there are more holomorphy types, one for each divisor of n.

To prevent the tables from becoming cumbersome, we have chosen to focus on just one distinction: whether or not the difference d is invertible (has maximum order) or singular (has order a proper divisor of n). In the first case, the progression is holomorphically equivalent to the progression $0, 1, 2, 3, \ldots, k$ where $k+1$ is the length of the progression. In the second case, the progression cannot be put into this form and will be called *singular*. It is interesting to note that sometimes all the maximum length progressions are singular. This is true, for example, of the two minimum (strict) sum covers modulo 30.

In the summary tables in the next section, columns 6, 7, and 8 give information on the longest arithmetic progressions in a sum cover. Column 6 gives the maximum (and hence the longest possible arithmetic progression in a minimal sum cover of $\mathbb{Z}_n$), and column 7 gives the smallest length of a maximum progression in a minimal sum cover of $\mathbb{Z}_n$. Column 8 gives the number of holomorphy classes in which there is a maximum length progression whose constant difference is singular—that is, has order a proper divisor of n.

In the case of the non-strict sum covers, we will also be measuring how far such a cover is from being strict. The *sum deficit set* of a set S is the set $A \setminus (S+S)$ of elements of A not obtained as sums from S. The *sum deficit* of S is the cardinality of the deficit set—that is,

$$\mathrm{def(S)} := |A \setminus (S+S)|$$

Likewise, the *strict sum deficit set* of a set S is the set $A \setminus (S\# + \#S)$ of elements of A not obtained as strict sums from S. The *strict deficit* of S is the cardinality of the strict deficit set—that is,

$$\mathrm{DEF}(S) := |A \setminus (S\# + \#S)|.$$

As noted in the section on generating slopes, in the hyperbolic and parabolic cases, some extra points must always be added to obtain some missing slopes. This means that, if the strict deficit of a non-strict cover is small enough, there is a chance the extra points can be arranged to cover the missing slopes.

The strict deficit represents one kind of gap that is of interest. Another gap of interest is the gap $\mathrm{MSC}(n) - \mathrm{CLB}(n)$ between the minimum size of a strict cover and the counting lower bound $\mathrm{CLB}(n)$ established in Theorem 1.1. A third gap of interest is the gap $\mathrm{MSC}(n) - \mathrm{msc}(n)$ between the minimum size of a strict cover and the minimum size of a non-strict cover. All this data is reported in the tables in the next section.

5 The tables

Below are summary tables for the census of all strict and non-strict sum covers for $\mathbb{Z}_n$ for $n \leq 54$.

The table for strict sum covers contains the following information:
Column 1 contains the value of n, the order of the cyclic group.
Column 2 contains the value of $\mathrm{MSC}(n)$, the minimum cardinality of a strict sum cover of $\mathbb{Z}_n$.
Column 3 contains the gap $\mathrm{MSC}(n) - \mathrm{CLB}(n)$ between the minimum size of a strict cover and the counting lower bound $\mathrm{CLB}(n) = \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil$.
Column 4 contains the gap $\mathrm{MSC}(n) - \mathrm{msc}(n)$ between the minimum sizes of a strict and non-strict covers.

Column 5 contains the number of holomorphy equivalence classes of strict sum covers. (The program was set to terminate after 500 classes were found.)
Column 6 contains the maximum length of a progression in a sum cover.
Column 7 contains the minimum value of the maximum length of a progression in a minimal sum cover.
Column 8 contains the number of holomorphy classes in which there is a maximum length progression with a singular constant difference.

The organization of the columns for the non-strict covers is similar except: Column 2 gives the maximum value of the strict deficit over all minimum sum covers, and Column 3 gives the minimum value of the deficit.

Several features of the tables are worth pointing out. First, there are five deviations from monotonicity occurring for $n = 23, 41, 51$ for MSC and for $n = 18, 28$ for msc. There are several cases in which MSC and msc agree: $n = 22, 24, 28, 36, 50, 52, 53$. There are certain "thresholds" at which the minimum size cover jumps by one. For the counting lower bound CLB, these thresholds are at the triangle numbers $\binom{m}{2}$. For MSC and msc they come a bit sooner and are less clearly defined because of the lack of monotonicity. As one would expect, when a threshold is passed and an extra point becomes necessary for a cover, there is much greater freedom in choosing the cover and consequently the number of equivalences classes increases dramatically. The number of classes then slowly declines until the next threshold. The table also suggests that the gap between the counting lower bound CLB and MSC tends to grow (although rather slowly) as n grows. This is problematical because the best known general constructions are close to the base b construction of Theorem (1.2). Thus there is likely an expanding gap between the best covers that we can produce and the actual minimum as n increases.

For a prime modulus n, all nonzero elements are invertible, so of course there are no singular maximum length progressions for prime moduli. Most of the composite moduli do exhibit minimum sum covers with singular maximum length progressions. However, when the number of sum covers is small there are exceptions: $n = 4, 6, 9, 22, 36$ in the strict case and $n = 4, 6, 8, 9, 10, 30, 34$ in the non-strict case. On the other hand as noted in Section 4, sometimes all the maximum length progressions are singular as occurs for both minimum (strict) sum covers modulo 30.

Following the summary tables, we give for each modulus the lexicographically first strict sum cover (and hence the one with the longest nonsingular arithmetic progression) and a non-strict sum cover of minimum strict deficit.

Strict Covers

Mod n	MSC	MSC-CLB	MSC-msc	Equivalence Classes	Progressions		
					max	min	Singular
3	3	0	1	1	3	3	0
4	4	0	1	1	4	4	0
5	4	0	1	1	4	4	0
6	4	0	0	1	3	3	0
7	5	0	1	1	5	5	0
8	5	0	1	2	4	4	1
9	5	0	1	1	4	4	0
10	6	1	1	6	5	4	2
11	6	0	1	2	5	4	0
12	6	0	1	6	5	3	2
13	6	0	1	1	4	4	0
14	7	1	1	14	6	4	5
15	7	1	1	13	6	3	2
16	7	0	1	7	5	4	4
17	7	0	1	1	5	5	0
18	8	1	1	79	7	3	31
19	8	1	2	18	6	4	0
20	8	1	1	40	6	4	20
21	8	1	1	8	6	4	1
22	8	0	0	1	5	5	0
23	9	1	1	94	7	4	0
24	8	0	0	1	4	4	1
25	9	1	1	41	7	3	5
26	9	1	1	22	6	4	9
27	9	1	1	6	6	3	1
28	9	1	0	1	4	4	1

Strict Covers

Mod n	MSC	MSC-CLB	MSC-msc	Equivalence Classes	Progressions		
					max	min	Singular
29	9	0	1	1	4	4	0
30	9	0	1	2	5	4	2
31	10	1	1	53	7	3	0
32	10	1	1	39	7	3	12
33	10	1	1	23	6	3	7
34	10	1	1	5	6	4	2
35	10	1	1	4	6	4	2
36	10	1	0	1	4	4	0
37	11	1	1	182	8	4	0
38	11	1	1	153	7	3	51
39	11	1	1	61	7	3	17
40	11	1	1	24	7	3	10
41	12	2	2	≥ 500	9	6	0
42	11	1	1	17	6	4	14
43	12	2	2	≥ 500	8	5	0
44	12	2	1	≥ 500	8	4	66
45	12	2	1	328	8	3	37
46	12	1	1	139	8	3	43
47	12	1	1	25	7	3	0
48	12	1	1	84	7	3	59
49	12	1	1	8	6	4	4
50	12	1	0	6	6	4	4
51	13	2	1	≥ 500	9	6	1
52	12	1	0	2	4	4	2
53	12	1	0	1	4	4	0
54	13	2	1	≥ 500	8	3	257

Non-Strict Covers

Mod n	msc	Deficits max	Deficits min	Equivalence Classes	Progressions max	Progressions min	Progressions Singular
3	2	2	2	1	2	2	0
4	3	1	1	1	3	3	0
5	3	2	2	1	3	3	0
6	4	2	0	3	4	2	0
7	4	2	1	2	4	3	0
8	4	3	2	2	3	2	0
9	4	4	4	1	2	2	0
10	5	2	1	3	4	3	0
11	5	3	2	2	4	3	0
12	5	3	3	1	3	3	1
13	5	4	4	1	3	3	0
14	6	3	1	8	5	3	3
15	6	4	2	12	4	3	6
16	6	4	2	3	4	2	1
17	6	4	4	1	4	4	0
18	7	4	1	24	5	3	7
19	6	6	6	1	2	2	0
20	7	4	1	6	5	3	2
21	7	5	2	6	5	3	0
22	8	5	0	97	6	3	33
23	8	6	1	61	6	3	0
24	8	6	0	71	6	2	24
25	8	6	2	23	6	3	1
26	8	7	2	7	5	2	1
27	8	8	4	10	4	2	1
28	9	5	0	256	7	3	84

Non-Strict Covers

Mod	msc	Deficits		Equivalence	Progressions		
n		max	min	Classes	max	min	Singular
29	8	6	6	1	3	3	0
30	8	6	6	1	2	2	0
31	9	7	2	30	6	3	0
32	9	6	3	8	5	3	2
33	9	7	2	13	5	3	6
34	9	6	6	1	3	3	0
35	9	8	8	2	3	3	0
36	10	7	0	280	7	3	113
37	10	10	1	115	6	2	0
38	10	7	2	36	6	3	12
39	10	8	2	65	6	3	20
40	10	7	2	23	5	2	16
41	10	8	4	5	4	3	0
42	10	7	2	5	4	3	3
43	10	8	6	2	4	3	0
44	11	8	1	161	7	3	59
45	11	9	2	170	7	3	16
46	11	8	2	21	5	3	7
47	11	8	2	18	6	3	0
48	11	8	2	15	5	3	14
49	11	6	4	2	5	5	0
50	12	8	0	≥ 500	8	4	57
51	12	8	1	≥ 500	8	4	8
52	12	9	0	327	7	3	153
53	12	10	0	172	6	3	0
54	12	10	1	50	7	2	17

Mod	Strict Non-Strict
3	$\{0,1,2\}$ $\{0,1\}$
4	$\{0,1,2,3\}$ $\{0,1,2\}$
5	$\{0,1,2,3\}$ $\{0,1,2\}$
6	$\{0,1,2,4\}$ $\{0,1,2,4\}$
7	$\{0,1,2,3,4\}$ $\{0,1,2,4\}$
8	$\{0,1,2,3,5\}$ $\{0,1,2,5\}$
9	$\{0,1,2,3,6\}$ $\{0,1,3,4\}$
10	$\{0,1,2,3,4,6\}$ $\{0,1,2,3,6\}$
11	$\{0,1,2,3,4,7\}$ $\{0,1,2,3,7\}$
12	$\{0,1,2,3,4,8\}$ $\{0,1,3,5,6\}$
13	$\{0,1,2,3,6,10\}$ $\{0,1,2,6,9\}$
14	$\{0,1,2,3,4,5,9\}$ $\{0,1,2,3,6,10\}$
15	$\{0,1,2,3,4,5,10\}$ $\{0,1,2,3,7,11\}$
16	$\{0,1,2,3,4,7,12\}$ $\{0,1,2,4,9,14\}$
17	$\{0,1,2,3,4,8,13\}$ $\{0,1,2,3,8,12\}$
18	$\{0,1,2,3,4,5,6,12\}$ $\{0,1,2,3,4,8,13\}$
19	$\{0,1,2,3,4,5,8,14\}$ $\{0,1,3,12,14,15\}$
20	$\{0,1,2,3,4,5,9,15\}$ $\{0,1,4,6,8,11,12\}$
21	$\{0,1,2,3,4,5,10,16\}$ $\{0,1,2,3,8,12,16\}$
22	$\{0,1,2,3,4,8,13,18\}$ $\{0,1,2,3,4,8,13,18\}$

Mod	Strict Non-Strict
23	$\{0,1,2,3,4,5,6,10,17\}$ $\{0,1,2,3,4,8,13,18\}$
24	$\{0,1,2,4,8,13,18,22\}$ $\{0,1,2,4,8,13,18,22\}$
25	$\{0,1,2,3,4,5,6,12,19\}$ $\{0,1,2,3,5,11,17,23\}$
26	$\{0,1,2,3,4,5,9,15,21\}$ $\{0,1,2,3,4,10,15,20\}$
27	$\{0,1,2,3,4,5,10,16,22\}$ $\{0,1,2,3,8,12,18,22\}$
28	$\{0,1,2,4,8,10,15,20,26\}$ $\{0,1,2,4,8,10,15,20,26\}$
29	$\{0,1,2,3,5,10,16,22,27\}$ $\{0,1,2,6,7,8,17,20\}$
30	$\{0,1,2,4,10,15,17,22,28\}$ $\{0,1,3,9,11,12,16,26\}$
31	$\{0,1,2,3,4,5,6,11,18,25\}$ $\{0,1,2,3,4,5,12,18,24\}$
32	$\{0,1,2,3,4,5,6,12,19,26\}$ $\{0,1,2,3,4,10,15,21,26\}$
33	$\{0,1,2,3,4,5,10,16,22,28\}$ $\{0,1,2,3,9,13,18,23,27\}$
34	$\{0,1,2,3,4,6,12,19,26,32\}$ $\{0,1,2,4,10,12,13,17,31\}$
35	$\{0,1,2,3,5,10,16,22,28,33\}$ $\{0,1,2,7,8,11,26,29,30\}$
36	$\{0,1,2,3,6,12,19,20,27,33\}$ $\{0,1,2,3,6,12,19,20,27,33\}$
37	$\{0,1,2,3,4,5,6,7,14,22,30\}$ $\{0,1,2,3,4,10,15,20,26,30\}$
38	$\{0,1,2,3,4,5,6,11,18,25,32\}$ $\{0,1,2,3,4,10,16,21,26,32\}$
39	$\{0,1,2,3,4,5,6,12,19,26,33\}$ $\{0,1,2,3,6,11,18,24,31,36\}$
40	$\{0,1,2,3,4,5,10,17,23,29,33\}$ $\{0,1,2,3,4,11,16,22,28,33\}$
41	$\{0,1,2,3,4,5,6,7,8,15,24,33\}$ $\{0,1,2,3,10,14,19,25,30,34\}$
42	$\{0,1,2,3,4,9,16,20,26,30,37\}$ $\{0,1,3,6,8,9,18,22,29,33\}$

Mod	Strict Non-Strict
43	$\{0,1,2,3,4,5,6,7,12,20,28,36\}$ $\{0,1,2,6,7,8,18,21,30,33\}$
44	$\{0,1,2,3,4,5,6,7,13,21,29,37\}$ $\{0,1,2,3,4,5,12,18,24,31,36\}$
45	$\{0,1,2,3,4,5,6,7,14,22,30,38\}$ $\{0,1,2,3,4,5,12,19,25,31,38\}$
46	$\{0,1,2,3,4,5,6,12,19,26,33,40\}$ $\{0,1,2,3,4,10,17,22,28,33,40\}$
47	$\{0,1,2,3,4,5,6,12,20,27,34,39\}$ $\{0,1,2,3,4,5,13,19,26,33,39\}$
48	$\{0,1,2,3,4,5,6,13,21,27,33,41\}$ $\{0,1,2,6,8,17,20,30,33,42,44\}$
49	$\{0,1,2,3,4,5,11,18,23,31,36,43\}$ $\{0,1,2,3,4,12,17,23,30,36,41\}$
50	$\{0,1,2,3,4,5,11,19,24,31,36,44\}$ $\{0,1,2,3,4,5,11,19,24,31,36,44\}$
51	$\{0,1,2,3,4,5,6,7,8,16,26,33,43\}$ $\{0,1,2,3,4,5,6,14,21,28,36,42\}$
52	$\{0,1,2,4,5,6,11,19,27,31,39,47\}$ $\{0,1,2,4,5,6,11,19,27,31,39,47\}$
53	$\{0,1,2,3,6,11,18,25,31,38,45,50\}$ $\{0,1,2,3,6,11,18,25,31,38,45,50\}$
54	$\{0,1,2,3,4,5,6,7,14,23,31,39,45\}$ $\{0,1,2,4,6,12,19,28,32,37,44,52\}$

References

[1] Clark, Edwin and Robert E. Jamison, Generating all slopes in AG$(2, q)$ with a minimum number of points, in preparation.

[2] Fisher, J. Chris and Robert E. Jamison, Properties of affinely regular polygons, *Geometriae Dedicata* **69** (1998), 241–259.

[3] Graham, R. L. and N. J. A. Sloane, On additive bases and harmonious graphs. *SIAM J. Algebraic Discrete Methods* **1** (1980), no. 4, 382–404.

[4] Halberstam, H. and K. F. Roth, *Sequences*, Springer-Verlag, Berlin, 1983.

[5] Jamison, Robert E., A survey of the slope problem in Discrete Geometry and Convexity, *Annals of the N.Y. Acad. Sci.* **440** (1985), 34–51.

[6] Jamison, Robert E., The Helly bound for singular sums, in preparation.

[7] Jamison, Robert E., Singular sum covers of factored Abelian groups, in preparation.

[8] Nathanson, Melvyn B., *Additive Number Theory: The Classical Bases*, Graduate Texts in Mathematics 164, Springer-Verlag, Berlin, 1996.