Introduction to Abstract Algebra
Final Exam

Instructions

1. Do NOT write your answers on these sheets. Nothing written on the test papers will be graded.
2. Do NOT write your name on any of your answer sheets.
3. Please begin each section of questions on a new sheet of paper.
4. Do not write problems side by side.
5. Do not staple test papers.
6. Limited credit will be given for incomplete or incorrect justification.

Questions

1. Simple Reason (3 each)
   Determine whether the following statements are true or false. If false give a counterexample. If true state the definition or theorem supporting it.

   (a) $(\mathbb{Z}, \times)$ is not a group.
   True. 0,2,-5 and every element other than 1,-1 do not have an inverse.

   (b) There is a subgroup of some S_n of order 7.
   True. The rotations of seven points (half of D_{14}).

   (c) $\mathbb{Z}_{121}$ is not an integral domain.
   True. $121 = 11 \cdot 11$ so there are zero divisors.

   (d) S_n always has a cyclic subgroup of order n.
   True. The rotations of n points (half of D_{2n}).

   (e) The set of all elements of the Q_8 such that $x^4 = 1$ is a group.
   True. This is the whole group.

   (f) Q_8 has a non-trivial normal subgroup.
   True. The group $\{1, -1\}$.

   (g) $\mathbb{Z}_n$ has a normal subgroup for all n.
   True. $\mathbb{Z}_n$ is abelian.

   (h) There is a group with subgroups of order 1,2,3,5,7.
   True. Cyclic groups have subgroups of all orders dividing the order. Consider Z_n for $n = 2 \cdot 3 \cdot 5 \cdot 7$.

   (i) S_n always has subgroups of order $1, 2, \ldots, n$.
   True. The subgroups generated by $(12 \ldots n)$ generate a cyclic subgroup

   (j) There exists a field such that $(F, +)$ and $(F^*, \times)$ are both cyclic.
   True. $\mathbb{Z}_{13}$.

2. Calculations (3 each)

   (a) Write the inverse element of (125)(34) in S_5.

$$(152)(34)$$

   (b) Find the order of (125)(34) in S_5.
   The order of (125) is 3 and the order of (34) is 2. The order of this element is 6.

   (c) Write all the left cosets of $\{e, (12)\}$ in S_3.

$$\begin{aligned} eH &= \{e, (12)\}. \\ (13)H &= \{(13), (132)\}. \\ (23)H &+ \{(23), (123)\}. \end{aligned}$$

   (d) Multiply the Hamiltonian quaternions $(1 + 4i - 4j + 5k)(-1 + 3i - 3j + 6k)$

$$\begin{aligned} (1 + 4i - 4j + 5k)(-1 + 3i - 3j + 6k) &= -1 + 3i - 3j + 6k + \\ &\quad -4i - 12 - 12k - 24j + \\ &\quad +4j + 12k - 12 - 24i + \\ &\quad -5k + 15j + 15i - 30 \\ &= -55 - 10i - 8j + k. \end{aligned}$$

   (e) Using the quotient group in Figure 2, find $(iH)(jH)$.

$$\begin{aligned} (iH)(jH) &= (ij)H \text{ by def.} \\ &= kH. \end{aligned}$$

   (f) Name a group isomorphic to the quotient group in Figure 2.
   Every element is its own inverse, so this is K_4.

   (g) $\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic. State the generator. Give the isomorphism between this group and $\mathbb{Z}_6$.
   The generator is $(1, 1)$. An isomorphism is $f(n(1, 1)) \mapsto n1$ (generator to generator).

3. Proof

(a) In a quotient ring the multiplicative inverses are unique.

In a quotient ring R the elements R^* form a group under multiplication. Thus uniqueness of inverse is known.

(b) The intersection of two subgroups is a subgroup.

Consider $G_1 \cap G_2$.

Note $G_1 \cap G_2 \subseteq G_1$ by previous theorem.

Closure: Let $a, b \in G_1 \cap G_2$. Thus $a, b \in G_1$ and $a, b \in G_2$ by definition of intersection. By closure in groups $ab \in G_1$ and $ab \in G_2$. Thus $ab \in G_1 \cap G_2$.

Inverse: Let $a \in G_1 \cap G_2$. Thus $a \in G_1$ and $a \in G_2$ by definition of intersection. By definition of groups $a^{-1} \in G_1$ and $a^{-1} \in G_2$. Thus $a^{-1} \in G_1 \cap G_2$.

Therefore $G_1 \cap G_2 \leq G_1$.

(c) If $a^2 = e$ for all $a \in G$ then G is Abelian.

Note, because $a^2 = e$ by definition of inverse $a = a^{-1}$. Consider

$$\begin{aligned} ab &= (ab)^{-1} \text{ given} \\ &= b^{-1}a^{-1} \text{ previous theorem} \\ &= ba \text{ given.} \end{aligned}$$

	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

Figure 1: Quaternions (Q_8)

	1	i	j	k
1	1	i	j	k
-1	-1	$-i$	$-j$	$-k$

Figure 2: Quotient Group of Quaternions