Introduction to Abstract Algebra
Exam 3 Key

Instructions

1. Do NOT write your answers on these sheets. Nothing written on the test papers will be graded.
2. Do NOT write your name on any of your answer sheets.
3. Please begin each section of questions on a new sheet of paper.
4. Do not write problems side by side.
5. Do not staple test papers.
6. Limited credit will be given for incomplete or incorrect justification.

Questions

1. Homomorphisms
   (a) (3) Calculate the Ker(f) for $f : \mathbb{Z}_{24} \to \mathbb{Z}_8$ defined by $f(n) = n \bmod 8$.

   $$\begin{aligned} f(n) &= 0. \\ n \bmod 8 &= 0. \\ 8 &\mid n. \end{aligned}$$

   $$\text{Ker}(f) = \{0, 8, 16\}.$$

   (b) (3) How many elements are in $f^{-1}(1)$?
   Based on a theorem f^{-1} returns a coset of the kernel, so there are three elemets.

   (c) (3) Write all the ordered pairs for the inner automorphism $g : Q_8 \to Q_8$ defined by $g(x) = -ixi$.

x	1	-1	i	$-i$	j	$-j$	k	$-k$
$g(x)$	1	-1	i	$-i$	$-j$	j	$-k$	k

   (d) (3) Demonstrate that $g(jk) = g(j)g(k)$ for g above.

   $$\begin{aligned} g(jk) &= \\ g(i) &= i \\ &= (-j)(-k) \\ &= g(j)g(k). \end{aligned}$$

2. Normal Groups

   (a) (3) Find a normal subgroup of $\mathbb{Z}_6$ (neither trivial case is acceptable).
   $\mathbb{Z}_6$ is Abelian so all subgroups are normal. $H = \{0, 3\}$ is one.

   (b) (3) How many non-trivial normal subgroups does Q_8 have?
   $\{1, -1\}$ is the center, so it is normal.
   $\{1, -1, i, -i\}$, $\{1, -1, j, -j\}$, $\{1, -1, k, -k\}$ have only two cosets, so they are normal.
   There are no other subgroups.

   (c) (3) How many non-trivial normal subgroups does $\mathbb{Z}_{11}$ have?
   $\mathbb{Z}_1 1$ has no non-trivial subgroups, because it is of prime order.

   (d) (3) Is $\mathbb{Z}_8/\{0, 2, 4, 6\}$ Abelian? The subgroup is normal.
   $\mathbb{Z}_8$ is Abelian so its quotient groups are Abelian.

3. Truth

(a) Determine if the following are true or false and provide a very brief explanation for each of the following. (3 each)

i. A homomorphism can map an Abelian group to a non-Abelian group.
Yes, because a homomorphism does not need to be onto. That is it can map an Abelian group onto an Abelian subgroup of a non-Abelian group.

ii. A group can be normal.
Normal refers to subgroups. However, every group is a subgroup of itself and is normal.

iii. A group can have a homomorphism from itself to itself that is not an automorphism.
Yes, because the homomorphism could be onto a subgroup (e.g., $f : \mathbb{Z}_8 \to \mathbb{Z}_4 \leq \mathbb{Z}_8$).

iv. Some group has no automorphism.
This is false, because every group has the identity automorphism.

v. All groups of order 4 are isomorphic.
False, K_4 is not cyclic but $\mathbb{Z}_4$ is.

vi. A group of order 7 can have 14 inner automorphisms.
False, inner automorphisms are defined by an element and there are only 7.

(b) (5) Prove that if a homomorphism $\text{Ker}(f) = \{e\}$ then f is one-to-one.

Proof: Suppose

$$\begin{aligned}
f(a) &= f(b). && \\
f(a)f(b)^{-1} &= f(b)f(b)^{-1}. && \text{Groups have inveses.} \\
f(a)f(b)^{-1} &= e. && \text{Def. inverse} \\
f(a)f(b^{-1}) &= e. && \text{Thm. homomorphisms} \\
f(ab^{-1}) &= e. && \text{Def. homomorphism} \\
ab^{-1} &= e. && \text{Ker}(f) = \{e\}. \\
ab^{-1}b &= eb. && \\
a &= b. && \text{Def. inverse}
\end{aligned}$$

Thus the homomorphism is 1-1.

Quaternions (Q_8)

	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1