Introduction to Abstract Algebra
Exam 2 Key

Instructions

1. Do NOT write your answers on these sheets. Nothing written on the test papers will be graded.
2. Do NOT write your name on any of your answer sheets.
3. Please begin each section of questions on a new sheet of paper.
4. Do not write problems side by side.
5. Do not staple test papers.
6. Limited credit will be given for incomplete or incorrect justification.

Questions

1. Groups

   (a) (4) Per a previous theorem the union of subgroups is rarely a subgroup. For the following subgroups illustrate why this is true. $H_1 = \{0, 2, 4\}$. $H_2 = \{0, 3\}$. $G = (\mathbb{Z}_6, +)$.
   Note $H_1 \cup H_2 = \{0, 2, 3, 4\}$. Also $2 + 3 = 5 \notin H_1 \cup H_2$. Thus this is not a subgroup due to lack of closure.

   (b) (5) Prove that $H = \{5n \bmod 200 : n \in \mathbb{Z}\}$ is a subgroup of $\mathbb{Z}_{200}$
   The multiples of five is a subgroup of $\mathbb{Z}_{200}$.
   Proof: Closure
   Let $a, b \in H$. Thus $a = 5n$, and $b = 5m$ for some $m, n \in \mathbb{Z}$. Thus

   $$\begin{aligned} a + b &= 5n + 5m \\ &= 5(n + m). \end{aligned}$$

   Note also that $40[5(n + m) - 5k] = 40[5(n + m - k)] = 200(n + m - k)$. Thus $a + b$ is equivalent to some multiple of 5 mod 200. H is closed.
   Proof: Inverse
   let $a \in H$. Thus $a = 5n$ for some $n \in \mathbb{Z}$. Note

   $$\begin{aligned} 200 - 5n &= \\ 5(40 - n). \end{aligned}$$

   Note also that $40[5(40 - n) - 5k] = 40[5(40 - n - k)] = 200(40 - n - k)$. Also $5n + (200 - 5n) = 0$. Thus a^{-1} is equivalent to some multiple of 5 mod 200. H has inverses.
   Thus $H \leq G$. □

   (c) (3) Prove or disprove that D_8 and Q_8 are isomorphic.
   D_8 has 6 elements that are their own inverses. Q_8 does not. They are not isomorphic.

   (d) (3) Find a cyclic subgroup of D_8 of cardinality greater than 2.
   $H = \{e, (1234), (13)(24), (1432)\}$.

2. Cosets

   (a) (3) Note $St(3) \leq S_4$. Write two, distinct cosets of $St(3)$.

$$
\begin{aligned}
eSt(3) &= St(3). \\
(13)St(3) &= \{(13), (132), (134), (13)(24), (1324), (1342)\}. \\
(23)St(3) &= \{(23), (123), (14)(23), (234), (1234), (1423)\}. \\
(34)St(3) &= \{(34), (12)(34), (143), (243), (1243), (1432)\}.
\end{aligned}
$$

   (b) (3) How many cosets does $St(3)$ have in S_4?

$$\frac{|S_4|}{|St(3)|} = \frac{24}{6} = 4.$$

   (c) (3) Select the subgroup of $\mathbb{Z}_5$ that produces the maximum number of cosets (largest index) other than $\{e\}$.
   $|Z_5| = 5$. 5 is prime so the only subgroups are $\{e\}$ (excluded) and $\mathbb{Z}_5$.

3. Let it Proof (5 each)

   (a) The set $H = \{x \in G | x^n = e\}$ for fixed $n \in \mathbb{Z}$ is a subgroup of an Abelian group G.
   Proof: Closure
   Let $a, b \in H$. Note

$$\begin{aligned}(ab)^n &= a^n b^n \text{ Abelian} \\ &= ee \text{ given} \\ &= e \text{ def. of identity}\end{aligned}$$

   Thus the set is closed.
   Proof: Inverses
   Let $a \in H$. Note

$$\begin{aligned}(a^{-1})^n &= (a^n)^{-1} \text{ previous theorem} \\ &= e^{-1} \text{ given} \\ &= e \text{ def. of identity}\end{aligned}$$

   Thus $a^{-1} \in H$.
   This set is a subgroup. □

   (b) The center $Z = \{x \in G | xg = gx \ \ \forall g \in G\}$ is a subgroup of G.
   Proof: Closure
   Let $x, y \in Z$. Note

$$\begin{aligned}(xy)g &= x(yg) \text{ associative} \\ &= x(gy) \text{ given} \\ &= (xg)y \text{ associative} \\ &= (gx)y \text{ given} \\ &= g(xy) \text{ associative}\end{aligned}$$

   Thus Z is closed.
   Proof: Inverses
   Let $x \in Z$. Note

$$\begin{aligned}gx^{-1} &= (xg^{-1})^{-1} \text{ inverse thm.} \\ &= (g^{-1}x)^{-1} \text{ given} \\ &= x^{-1}g \text{ inverse thm.}\end{aligned}$$

   Thus Z contains inverses.
   Z is a subgroup. □

$$
\begin{aligned}
St(3) \quad = \quad & \{e, \\
& (12),(14),(24) \\
& (124),(142)\}
\end{aligned}
$$

Quaternions (Q_8)

	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

Dihedral 8

	e	(13)	(24)	(12)(34)	(13)(24)	(14)(23)	(1234)	(1432)
e	e	(13)	(24)	(12)(34)	(13)(24)	(14)(23)	(1234)	(1432)
(13)	(13)	e	(13)(24)	(1432)	(24)	(1234)	(14)(23)	(12)(34)
(24)	(24)	(13)(24)	e	(1234)	(13)	(1432)	(12)(34)	(14)(23)
(12)(34)	(12)(34)	(1234)	(1432)	e	(14)(23)	(13)(24)	(13)	(24)
(13)(24)	(13)(24)	(24)	(13)	(14)(23)	e	(12)(34)	(1432)	(1234)
(14)(23)	(14)(23)	(1432)	(1234)	(13)(24)	(12)(34)	e	(24)	(13)
(1234)	(1234)	(12)(34)	(14)(23)	(24)	(1432)	(13)	(13)(24)	e
(1432)	(1432)	(14)(23)	(12)(34)	(13)	(1234)	(24)	e	(13)(24)