Math 321
Test 3

Instructions

1. Do NOT write your answers on these sheets. Nothing written on the test papers will be graded.
2. Please begin each section of questions on a new sheet of paper.
3. Do not write answers side by side.
4. Please do not staple your test papers together.
5. Limited credit will be given for incomplete or incorrect justification.
6. Proofs must be written in the proper format.

1. Line Integrals (4 each)

Evaluate the specified scalar and vector line integrals.

(a) \( f(x, y, z) = x^2 - y^2 + z^2 \). \( \mathbf{x}(t) = [\cos t, t, \sin t] \), \( t \in [0, 2\pi] \).

\[
\int f \, ds = \int_{\mathbf{x}} f(\mathbf{x}(t))|\mathbf{x}'(t)| \, dt \\
= \int_{0}^{2\pi} (\cos^2 t - t^2 + \sin^2 t) \sqrt{(-\sin t)^2 + 1^2 + \cos^2 t} \, dt \\
= \int_{0}^{2\pi} (1 - t^2) \sqrt{2} \, dt \\
= \sqrt{2} \left( t - \frac{1}{3}t^3 \right)^{2\pi}_{0} \\
= \sqrt{2} \left( 2\pi - \frac{8}{3}\pi^3 \right).
\]

(b) \( \mathbf{F}(x, y) = [2xy, x + y] \). \( \mathbf{x}(t) = [t, t^2], t \in [0, 5] \).

\[
\int_{\mathbf{x}} \mathbf{F} \cdot \, ds = \int_{\mathbf{x}} \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) \, dt \\
= \int_{0}^{5} [2(t)(t^2), t + t^2] \cdot [1, 2t] \, dt \\
= \int_{0}^{5} 2t^3 + (2t^2 + 2t^3) \, dt \\
= \int_{0}^{5} 4t^3 + 2t^2 \, dt \\
= t^4 + \frac{2}{3}t^3 \bigg|_{0}^{5} \\
= 5^4 + \frac{2}{3} \cdot 5^3 \\
= 625 + \frac{250}{3}.
\]
2. Green’s Theorem

(a) (2) State Green’s Theorem.

Let $D$ be a closed, bounded region in $\mathbb{R}^2$ whose boundary $C = \delta D$ consists of finitely many simple, closed curves. Orient the curves of $C$ so that $D$ is on the left as one traverses $C$. Let $F(x, y) = [M(x,y), N(x,y)]$ be a vector field of class $C^1$ throughout $D$. Then

$$\oint_C M\,dx + N\,dy = \int\int_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \,dxdy.$$ 

(b) (8) Use Green’s theorem to evaluate the following line integral two ways. $D$ is the polygonal region defined by the four vertices (-2,0), (0,0), (2,2), (0,2). The vector field is $F(x, y) = [\cos x, \sin y]$.

First parameterize the boundary of the region. We will need the derivatives of the boundary functions as well.

$C_1(t) = [-2+2t, 0] \quad t \in [0, 1] \quad C_1'(t) = [2, 0]$  
$C_2(t) = [2t, 2t] \quad t \in [0, 1] \quad C_2'(t) = [2, 2]$  
$C_3(t) = [2 - 2t, 2] \quad t \in [0, 1] \quad C_3'(t) = [-2, 0]$  
$C_4(t) = [-2t, 2 - 2t] \quad t \in [0, 1] \quad C_4'(t) = [-2, -2]$  

$$\oint_C F \cdot ds = \int_0^1 [\cos(-2 + 2t), \sin 0] \cdot [2, 0] \,dt + \int_0^1 [\cos(2t), \sin(2t)] \cdot [2, 2] \,dt + \int_0^1 [\cos(2 - 2t), \sin(2)] \cdot [-2, 0] \,dt + \int_0^1 [\cos(-2t), \sin(2 - 2t)] \cdot [-2, -2] \,dt$$

$$= \int_0^1 2 \cos(-2 + 2t) \,dt + \int_0^1 2 \cos(2t) + 2 \sin(2t) \,dt + \int_0^1 -2 \cos(2 - 2t) \,dt + \int_0^1 -2 \cos(-2t) - 2 \sin(2 - 2t) \,dt$$

$$= \int_0^1 2 \sin(2t) - 2 \sin(2 - 2t) \,dt = - \cos(2t) - \cos(2 - 2t)|_0^1 = (\cos 2 - \cos 0) - (\cos 0 - \cos 2) = 0.$$ 

We used $\cos x = \cos(-x)$ and $-\sin x = \sin(-x)$.

Now we calculate the area integral.

$$\int\int_D \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \,dA = \int_0^1 \int_{-2-x}^{2-x} 0 \,dy \,dx + \int_0^2 \int_x^{2-x} 0 \,dy \,dx = 0.$$
3. Conservative Vector Fields

(2+4) Determine if the following vector fields are conservative. If a vector field is conservative, find the scalar potential function.

(a) \( \mathbf{F}(x, y) = [x^2y, x^2 + y] \).

Because this is a vector field on \( \mathbb{R}^2 \) we can use the simplified condition \( \frac{\delta N}{\delta x} = \frac{\delta M}{\delta y} \).

\( \frac{\delta N}{\delta x} = 2x, \) and \( \frac{\delta M}{\delta y} = x^2. \) These are not equal; therefore, this vector field is not conservative.

(b) \( \mathbf{F}(x, y, z) = [yze^{xyz}, xze^{xyz}, xye^{xyz}] \).

We test the curl of \( \mathbf{F} \).

\[
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\
yze^{xyz} & xze^{xyz} & xye^{xyz}
\end{vmatrix}
= \begin{bmatrix}
\frac{\delta(yze^{xyz})}{\delta y} - \frac{\delta(xze^{xyz})}{\delta z} \\
\frac{\delta(xze^{xyz})}{\delta x} - \frac{\delta(yze^{xyz})}{\delta z} \\
\frac{\delta(xye^{xyz})}{\delta x} - \frac{\delta(yze^{xyz})}{\delta y}
\end{bmatrix} \mathbf{i} - \\
\begin{bmatrix}
\frac{\delta(yze^{xyz})}{\delta y} - \frac{\delta(xze^{xyz})}{\delta z} \\
\frac{\delta(xze^{xyz})}{\delta x} - \frac{\delta(yze^{xyz})}{\delta z} \\
\frac{\delta(xye^{xyz})}{\delta x} - \frac{\delta(yze^{xyz})}{\delta y}
\end{bmatrix} \mathbf{j} + \\
\begin{bmatrix}
\frac{\delta(yze^{xyz})}{\delta y} - \frac{\delta(xze^{xyz})}{\delta z} \\
\frac{\delta(xze^{xyz})}{\delta x} - \frac{\delta(yze^{xyz})}{\delta z} \\
\frac{\delta(xye^{xyz})}{\delta x} - \frac{\delta(yze^{xyz})}{\delta y}
\end{bmatrix} \mathbf{k}
\]

\[
= [(xy(xze^{xyz}) + xze^{xyz}) - (xz(xye^{xyz}) + xke^{xyz})] \mathbf{i} - \\
[(xy(yze^{xyz}) + yze^{xyz}) - (yz(xye^{xyz}) + ye^{xyz})] \mathbf{j} + \\
[(xz(yze^{xyz}) + zze^{xyz}) - (yz(xze^{xyz}) + ze^{xyz})] \mathbf{k}
\]

\[
= [(xyz + 1)xe^{xyz} - (xyz + 1)xe^{xyz}] \mathbf{i} - \\
[(xyz + 1)ye^{xyz} - (xyz + 1)ye^{xyz}] \mathbf{j} + \\
[(xyz + 1)ze^{xyz} - (xyz + 1)ze^{xyz}] \mathbf{k}
\]

\[
= 0.
\]

Thus this is a conservative vector field.

To find the scalar potential function we integrate as follows.

\[
f(x, y, z) = \int yze^{xyz} dx = e^{xyz} + g(y, z).
\]

To determine the function \( g(y, z) \) we differentiate \( f(x, y, z) \) with respect to \( y \) then \( z \).

\( \frac{\delta f}{\delta y} = xze^{xyz} + g'(y, z) = xze^{xyz}. \) Therefore \( g'(y, z) = 0, \) and \( g(y, z) = h(z). \)

\( \frac{\delta f}{\delta z} = xye^{xyz} + h'(z) = xye^{xyz}. \) Therefore \( h'(z) = 0, \) and \( h(z) = C. \)

Thus \( f(x, y, z) = e^{xyz} + C. \)
4. Theory (5)

Prove one of the following

(a) Let \( x : [a, b] \to \mathbb{R}^n \) be a piecewise \( C^1 \) path, and let \( f : \mathbb{R}^n \to \mathbb{R} \) be a continuous function whose domain contains the image of \( x \). If \( y : [c, d] \to \mathbb{R}^n \) is any reparametrization of \( x \), then

\[
\int_y f \, ds = \int_x f \, ds.
\]

(b) If \( D \) is a region to which Green’s theorem applies, and \( \mathbf{F} \) is a class \( C^1 \) vector fields on \( D \), then, orienting \( \delta D \) appropriately,

\[
\oint_{\delta D} \mathbf{F} \cdot d\mathbf{s} = \iint_D \text{curl}(F) \cdot \mathbf{k} \, dA.
\]