Math 321  
Test 1

Instructions

1. Do NOT write your answers on these sheets. Nothing written on the test papers will be graded.
2. Do NOT use staples.
3. Please begin each section of questions on a new sheet of paper.
4. Limited credit will be given for incomplete or incorrect justification.
5. Remember to use proper form for proofs.
6. For the take home portion, you may use the Colley textbook, but no other text book and no person.

1. Geometry

(a) (2) Give the equation of the line through the points (-1,2,5) and (7,3,2).
The vector from the first to the second point is: [8, 1, -3]. Therefore the equation of the line is given by
\[ \mathbf{x} = [8, 1, -3]t + [-1, 2, 5]. \]

(b) (3) Give the equation of the plane tangent to \( f(x, y) = x^2y + xy^2 \) at \( \mathbf{a} = (2, 3) \).
First calculate the partials \( f_x(x, y) = 2xy + y^2 \), and \( f_y(x, y) = x^2 + 2xy \). We need \( f_x(2, 3) = 21 \), and \( f_y(2, 3) = 16 \). Also \( f(2, 3) = 30 \). Thus the equation of the plane tangent to the curve \( f(x, y) \) at the point \( (2, 3) \) is
\[ z = 30 + 21(x - 2) + 16(y - 3). \]

(c) (3) Give the equation of the plane defined by the vectors \( [1, 1, 1], [-1, 2, -1] \) containing the point \( [1, 2, 3] \).
We need to find the normal vector for the plane. This is accomplished by calculating the cross product of the two vectors.

\[
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & 1 & 1 \\
-1 & 2 & -1 \\
\end{vmatrix} = -3\mathbf{i} - 0\mathbf{j} + 3\mathbf{k}
\]

Thus the equation for the plane is \(-3x + 0y + 3z + D = 0\). Using the point \([1, 2, 3]\) we calculate \(-3(1) + 0(2) + 3(3) + D = 0\), or \(D = -6\). Thus the equation for the plane is
\[-3x + 3z - 6 = 0.\]

(d) (1) Are the vectors \([1, 1, 1]\) and \([-1, 2, -1]\) orthogonal?
The dot products is \((1)(-1) + (1)(2) + (1)(-1) = -1 + 2 - 1 = 0\). Therefore the vectors are orthogonal.
2. Proofs
Prove two of the following (4 each).

(a) \( \mathbf{a} \cdot \mathbf{a} \geq 0 \).
   Proof:
   Let \( \mathbf{a} = [a_1, a_2, \ldots, a_n] \). \( \mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + \ldots + a_n^2 \) which is a sum of non-negative numbers. Therefore \( \mathbf{a} \cdot \mathbf{a} \geq 0 \) .

(b) Assuming \( f : \mathbb{R}^n \to \mathbb{R} \) and \( g : \mathbb{R}^n \to \mathbb{R} \) prove the following.
   If \( \lim_{x \to a} f(x) = L \), and \( \lim_{x \to a} g(x) = M \), then \( \lim_{x \to a} (f + g)(x) = L + M \).
   Proof:
   We show that for all \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that \( \|x - a\| < \delta \) implies \( \|f(x) - (L + M)\| < \epsilon \).
   Because \( \lim_{x \to a} f(x) = L \), for \( \epsilon_f = \frac{\epsilon}{2} \) there exists a \( \delta_f \) such that \( \|x - a\| < \delta_f \) implies \( \|f(x) - a\| < \frac{\epsilon}{2} \).
   Likewise, because \( \lim_{x \to a} g(x) = M \), for \( \epsilon_g = \frac{\epsilon}{2} \) there exists a \( \delta_g \) such that \( \|x - a\| < \delta_g \) implies \( \|g(x) - M\| < \frac{\epsilon}{2} \).
   Let \( \delta = \min(\delta_f, \delta_g) \). Thus if \( \|x - a\| < \delta \),

   \[
   \|(f + g)(x) - (L + M)\| = \|f(x) + g(x) - L - M\| \\
   = \|f(x) - L + g(x) - M\| \\
   \leq \|f(x) - L\| + \|g(x) - M\| \\
   < \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
   = \epsilon.
   \]

   Thus \( \lim_{x \to a} (f + g)(x) = L + M \). \( \Box \)

(c) If \( f \) and \( g \) are functions in \( C^1 \) (smooth functions), then \( D(f + g) = Df + Dg \).
   See \textit{Vector Calculus} by Colley pages 138-139.
3. Limits and Continuity

Determine if the following limits exist. If no, state why. Otherwise write “Probably.”

a. \( \lim_{(x,y,z) \to (0,0,0)} \frac{x^2 + y^2 + 3z^2}{x^2 + y^2 + z^2} \).

Note that if we fix \( y = z = 0 \), that is we take the limit along the \( x \)-axis it is

\[
\lim_{(x,0,0) \to (0,0,0)} \frac{x^2 + 0^2 + 3 \cdot 0^2}{x^2 + 0^2 + 0^2} = \lim_{(x,0,0) \to (0,0,0)} \frac{x^2}{x^2} = 1 \text{ by L’Hopital’s rule.}
\]

Note that if we fix \( x = z = 0 \), that is we take the limit along the \( y \)-axis it is

\[
\lim_{(0,y,0) \to (0,0,0)} \frac{0^2 + y^2 + 3 \cdot 0^2}{0^2 + y^2 + 0^2} = \lim_{(0,y,0) \to (0,0,0)} \frac{y^2}{y^2} = 1 \text{ by L’Hopital’s rule.}
\]

Note that if we fix \( x = y = 0 \), that is we take the limit along the \( z \)-axis it is

\[
\lim_{(0,0,z) \to (0,0,0)} \frac{0^2 + 0^2 + 3z^2}{0^2 + 0^2 + z^2} = \lim_{(0,0,z) \to (0,0,0)} \frac{3z^2}{z^2} = 3 \text{ by L’Hopital’s rule.}
\]

Thus the limit is not the same along all paths and does not exist.

b. \( \lim_{(x,y) \to (0,0)} [\cos(xy), \sin(xy)] \).

We determine the following component limits: \( \lim_{(x,y) \to (0,0)} \cos(xy) \), and \( \lim_{(x,y) \to (0,0)} \sin(xy) \). \( xy \) is a polynomial so it is continuous (the limit exists everywhere). \( \sin(x) \) and \( \cos(x) \) are continuous functions (the limit exists everywhere). The composition of continuous functions is continuous. Thus both limits exists, indeed we even know that the function is continuous.

Are the following functions continuous? If no, state why. Otherwise write “Probably”

c. \( f(x,y) = \frac{\sin(xy)}{x} \).

This is not continuous, because it is not defined at \((0,y)\) for any value \( y \). To be continuous a function must be defined.

d. \( g(x,y,z) = \frac{x+y+z}{e^x+e^y+e^z} \).

Note that \( x+y+z \) is a polynomial, and is continuous everywhere. Also \( \frac{1}{e^x+e^y+e^z} \) is continuous everywhere which we can prove. Thus the product is continuous.
4. Derivatives

(a) (3) Calculate $Df$ for $f(x, y, z) = [xe^z, ye^z, ze^z]$.

$Df = \begin{bmatrix} e^z & 0 & xe^z \\ 0 & e^z & ye^z \\ 0 & 0 & e^z + ze^z \end{bmatrix}.$

(b) (3) Calculate $Duf$ for $u = [1, 2, 3]$ and $f(x, y, z) = xy + yz + xz$.

$Df = [y + z, x + z, x + y]$.

We must find a unit vector in the direction of $u$. $u' = \frac{u}{\|u\|} = \frac{1}{\sqrt{14}}u$.

\[
Duf = Df \cdot u = \frac{1}{\sqrt{14}}[1, 2, 3] = \frac{1}{\sqrt{14}}(y + z) + \frac{2}{\sqrt{14}}(x + z) + \frac{3}{\sqrt{14}}(x + y) = \frac{5}{\sqrt{14}}x + \frac{4}{\sqrt{14}}y + \frac{3}{\sqrt{14}}z.
\]
5. Take Home

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be defined by $f(x) = [f_1(x), f_2(x), \ldots, f_m(x)]$. If $\lim_{x \to a} f_i(x) = L_i$ for all $i$, then $\lim_{x \to a} f(x) = L$, where $L = [L_1, L_2, \ldots, L_m]$.

Note $\lim_{x \to a} f(x) = L$, where $L = [L_1, L_2, \ldots, L_m]$, iff for all $\epsilon > 0$ there exists a $\delta > 0$ such that $\|x - a\| < \delta$ implies $\|f(x) - L\| < \epsilon$. 