Math 321
Final Exam
Take Home Portion Key

Instructions

1. Do NOT write your answers on these sheets. Nothing written on the test papers will be graded.
2. Please begin each section of questions on a new sheet of paper.
3. Do not write answers side by side.
4. Please do not staple your test papers together.
5. Limited credit will be given for incomplete or incorrect justification.
6. For the take home portion, any text book may be used, but no personal assistance is allowed.

1. \( \mathbf{F}(x, y) = [y, x] \). The curve \( C \) is parameterized by the two paths
   \[ \mathbf{x}_1(t) = [t, \cos(t) - 1] \text{ and } \mathbf{x}_2(t) = [4\pi - t, 1 - \cos(t)] \]
   with \( t \in [0, 4\pi] \) for both paths. Calculate the line integral \( \int_C \mathbf{F} \cdot d\mathbf{s} \).

   We test first if the vector field is conservative. Because this is a 2D vector field we need to check if
   \[ \frac{\delta N}{\delta x} = \frac{\delta M}{\delta y} . \]
   \[ \frac{\delta N}{\delta x} = 1 \]
   \[ \frac{\delta M}{\delta y} = 1 \]
   \[ \frac{\delta N}{\delta x} = \frac{\delta M}{\delta y} . \]

   Thus the vector field is conservative.

   Next we consider the curve.
This appears to be a pair of simple, closed curves. The path \( x_1(t) \) is two periods of a cos curve that has been reflected vertically and shifted up one. This implies that it touches the axis at \( 2\pi \) and \( 4\pi \). The path \( x_2(t) \) is two periods of a cos curve that has been shifted down one. This implies that it touches the axis at \( 2\pi \) and \( 4\pi \). Thus the two paths do form two, simple, closed curves.

The line integral over a finite set of simple, closed curves of a conservative vector field is 0.
2. \( f(x, y) = x^4 - 4x^2 + y^4 - 4y^2 \). \( \mathbf{v} = [2, 3, 0] \). \( \mathbf{N} : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) gives the normal vector of the plane tangent to \( f \) at the point \((x, y)\). Find \( \min_{(x,y) \in \mathbb{R}^2} \| \text{proj}_v \mathbf{N}(x, y) \| \).

Note that the vector \( \mathbf{v} = [2, 3, 0] \) lies in the \( xy \)-plane. We are looking to find a point in the domain so that the projection of the normal vector of the tangent plane at that point onto the \( xy \)-plane is minimized. Because we are considering magnitude, the smallest is 0 (no negative magnitudes). This means that we are looking for tangent planes with a vertical normal vector. This occurs when the tangent plane is horizontal \( z = c \) for some constant \( c \). This implies that the both partial derivatives at those points are zero (consider the equation for a tangent plane). Thus we are looking for extreme points of the function.

\[
\frac{\delta f}{\delta x} = 4x^3 - 8x.
\]
\[
\frac{\delta f}{\delta y} = 4y^3 - 8y.
\]

\[
4x^3 - 8x = 0.
\]
\[
4y^3 - 8y = 0.
\]

Thus the nine critical points are \((-\sqrt{2}, -\sqrt{2})\), \((-\sqrt{2}, 0)\), \((-\sqrt{2}, \sqrt{2})\), \((0, -\sqrt{2})\), \((0, 0)\), \((0, \sqrt{2})\), \((\sqrt{2}, -\sqrt{2})\), \((\sqrt{2}, 0)\), \((\sqrt{2}, \sqrt{2})\). We do not care if these are max, min, or saddle points, because all will have horizontal tangent planes.

For the curious the points are min, saddle, min, saddle, max, saddle, min, saddle, min respectively.
3. Find a global maximum of \( f(x, y, z) = (x + y)^3 + xyz \) subject to the constraints
\[
\begin{align*}
g_1(x, y, z) &= x^2 + y^2 + z^2 = 16; \\
g_2(x, y, z) &= z = 0; \\
g_3(x, y, z) &= -4y + 4z = 0; \\
g_4(x, y, z) &= x^2 + 4y^2 + 8z^2 = 16.
\end{align*}
\]
A constrained optimization problem asks us to find a min or max value of a function over a restricted domain. For this problem we consider the domain restrictions.

\( g_1 \) is a sphere with center at the origin and radius 4.
\( g_2 \) is a horizontal plane through the origin.
\( g_3 \) is a vertical plane through the origin.
\( g_4 \) is an ellipsoid with center at the origin.

Substituting \( z = 0 \) into \( g_1 \) and \( g_4 \) gives us
\[
\begin{align*}
x^2 + y^2 &= 16. \\
x^2 + 4y^2 &= 16.
\end{align*}
\]
\[
\begin{align*}
y^2 &= 16 - x^2. \\
4y^2 &= 16 - x^2.
\end{align*}
\]
\[
\begin{align*}
y^2 &= 4y^2. \\
3y^2 &= 0. \\
y &= 0.
\end{align*}
\]
For \( y = z = 0 \) \( g_3 \) is trivially satisfied. Substituting \( y = z = 0 \) into \( g_1 \) gives us \( x^2 = 16 \) or \( x = \pm 4 \). Thus the restricted domain is the two points \((4, 0, 0)\) and \((-4, 0, 0)\).

Plugging these two points into the function gives us \( f(4, 0, 0) = 4^3 = 64 \) and \( f(-4, 0, 0) = (-4)^3 = -64 \). Thus the maximum occurs at the domain point \((4, 0, 0)\).
4. Find the length of the curve given by the parameterization \( \mathbf{x}(t) = \left[ \frac{\cos t}{\sqrt{2}}, \sin t, \frac{\cos t}{\sqrt{2}} \right] \) \( t \in [0, 5\pi] \).

Graphing this function gives us the following which appears to be a circle.

A circle is a curve such that all points are equidistant from a point. The center would be the origin and the distance would be

\[
\| \mathbf{x}(t) \| = \sqrt{\left( \frac{\cos t}{\sqrt{2}} \right)^2 + (\sin t)^2 + \left( \frac{\cos t}{\sqrt{2}} \right)^2} \\
= \sqrt{\frac{\cos^2 t}{2} + \sin^2 t + \frac{\cos^2 t}{2}} \\
= \sqrt{\cos^2 t + \sin^2 t} \\
= \sqrt{1} \\
= 1.
\]

Thus every point is equidistant from the center, and the distance is 1.

The arclength of a circle is its circumference which is \( 2\pi \) in this case.
5. \( A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \) and \( c = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}. \) Consider the linear mapping \( \Gamma : \mathbb{R}^3 \to \mathbb{R}^3 \) given by \( \Gamma(\mathbf{x}) = B (A \mathbf{x} - \mathbf{c}). \) Find the rate of change of \( \Gamma \) at \([1, 1, 1]^T\).

A linear transformation is a function. In this case \( \Gamma : \mathbb{R}^3 \to \mathbb{R}^3. \) To find the rate of change (derivative) we rewrite this in our standard calculus form for a vector field.

\[
\Gamma \left( \begin{array}{c} x \\ y \\ z \end{array} \right) = B \left( \begin{array}{c} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{array} \right) \left( \begin{array}{c} x \\ y \\ z \end{array} \right) - c
\]

\[
= B \left( \begin{array}{c} x \\ 2y \\ x + y + z \end{array} \right) - \left( \begin{array}{c} 2 \\ 3 \\ 5 \end{array} \right)
\]

\[
= B \left( \begin{array}{c} x - 2 \\ 2y - 3 \\ x + y + z - 5 \end{array} \right)
\]

\[
= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x - 2 \\ 2y - 3 \\ x + y + z - 5 \end{bmatrix}
\]

\[
= \begin{bmatrix} x + y + z - 5 \\ 2y - 3 \\ x - 2 \end{bmatrix}.
\]

Now we calculate \( D\Gamma(\mathbf{x}) \).

\[
D\Gamma(\mathbf{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}
\]

Because the derivative is constant we are done.