Math 314
Final Exam Key

Instructions

1. Do NOT write your answers on these sheets. Nothing written on the test papers will be graded.
2. Please begin each section of questions on a new sheet of paper.
3. Do not write problems side by side.
4. Do not staple test papers.
5. Limited credit will be given for incomplete or incorrect justification.

Questions

$$A = \begin{bmatrix} 1 & -2 & 4 & -3 \\ 3 & 1 & 12 & 5 \\ 1 & -9 & 4 & -17 \\ 2 & 10 & 8 & 22 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\vec{A}_1 = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}, \vec{A}_2 = \begin{bmatrix} -2 & 1 \\ -9 & 10 \end{bmatrix}, \vec{A}_3 = \begin{bmatrix} 4 & 12 \\ 4 & 8 \end{bmatrix}, \vec{A}_4 = \begin{bmatrix} -3 & 5 \\ -17 & 22 \end{bmatrix}.$$

$$C = \begin{bmatrix} 0 & 2 & -1 \\ -1 & 0 & 2 \\ 2 & -1 & 0 \end{bmatrix},$$

$$T[x_1, x_2, x_3] = [2x_2 - x_3, 2x_3 - x_1, 2x_1 - x_2]^T.$$

1. Basic Bases

(a) (3) Give a basis for the column space of A.

$$\{[1, 3, 1, 2]^T, [-2, 1, -9, 10]^T\}$$

(b) (4) Give a basis for the null space of A.

$$\begin{aligned} x_1 + 4x_3 + x_4 &= 0. \\ x_1 &= -4x_3 - x_4. \\ x_2 + 2x_4 &= 0. \\ x_2 &= -2x_4. \\ [-4x_3 - x_4, -2x_4, x_3, x_4]^T &= x_3[-4, 0, 1, 0]^T + x_4[-1, -2, 0, 1]^T. \end{aligned}$$

$$\{[-4, 0, 1, 0]^T, [-1, -2, 0, 1]^T\}$$

(c) (3) List the dimension of the column space and null space of A.

$$\begin{aligned} \dim(\text{col})(A) &= 2. \\ \dim(\text{nul})(A) &= 2. \\ 2 + 2 &= 4. \end{aligned}$$

(d) (4) Identify a maximum, independent subset of A_1, A_2, A_3, A_4.

$$a \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} + b \begin{bmatrix} -2 & 1 \\ -9 & 10 \end{bmatrix} + c \begin{bmatrix} 4 & 12 \\ 4 & 8 \end{bmatrix} + d \begin{bmatrix} -3 & 5 \\ -17 & 22 \end{bmatrix} = \vec{0}.$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 4 & -3 & 0 \\ 3 & 1 & 12 & 5 & 0 \\ 1 & -9 & 4 & -17 & 0 \\ 2 & 10 & 8 & 22 & 0 \end{array}\right]$$

Note this is A, so an independent set is the last two.

2. Alternate Bases

Let $\mathcal{C} = \{[-1,2,0]^T, [0,-1,2]^T, [2,0,-1]^T\}$. Let $\mathcal{B}$ be the standard basis.

(a) (5) Find the $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ change of basis matrix.

$$\left[\begin{array}{rrrrrr} -1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \sim \quad R_2 \leftarrow 2R_1 + R_2$$

$$\left[\begin{array}{rrrrrr} -1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & 4 & 2 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \sim \quad R_3 \leftarrow 2R_2 + R_3$$

$$R_1 - 1 \leftarrow R_1$$

$$\left[\begin{array}{rrrrrr} -1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & 4 & 2 & 1 & 0 \\ 0 & 0 & 7 & 4 & 2 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{rrrrrr} 1 & 0 & -2 & -1 & 0 & 0 \\ 0 & -1 & 4 & 2 & 1 & 0 \\ 0 & 0 & 7 & 4 & 2 & 1 \end{array}\right] \sim \quad R_2 - 1 \leftarrow R_2$$

$$\left[\begin{array}{rrrrrr} 1 & 0 & -2 & -1 & 0 & 0 \\ 0 & 1 & -4 & -2 & -1 & 0 \\ 0 & 0 & 7 & 4 & 2 & 1 \end{array}\right] \sim \quad \begin{array}{l} R_1 \leftarrow \frac{2}{7}R_3 + R_1 \\ R_2 \leftarrow \frac{4}{7}R_3 + R_2 \\ R_3\frac{1}{7} \leftarrow R_3 \end{array}$$

$$\left[\begin{array}{rrrrrr} 1 & 0 & 0 & \frac{1}{7} & \frac{4}{7} & \frac{2}{7} \\ 0 & 1 & 0 & \frac{2}{7} & \frac{1}{7} & \frac{4}{7} \\ 0 & 0 & 1 & \frac{4}{7} & \frac{2}{7} & \frac{1}{7} \end{array}\right]$$

(b) (3) Find the inverse of the matrix above.

This is the $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ matrix which, because we start with the identity matrix is simply

$$\left[\begin{array}{rrr} -1 & 0 & 2 \\ 2 & -1 & 0 \\ 0 & 2 & -1 \end{array}\right]$$

(c) (3) Find the coordinate of $[7,5,2]^T$ with respect to $\mathcal{C}$.

Because $[7,5,2]_{\mathcal{B}} = [7,5,2]$, the coordinate with respect to $\mathcal{C}$ is

$$= \begin{bmatrix} \frac{1}{7} & \frac{4}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} & \frac{4}{7} \\ \frac{4}{7} & \frac{2}{7} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} 7 \\ 5 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} (\frac{1}{7})(7) + (\frac{4}{7})(5) + (\frac{2}{7})(2) \\ (\frac{2}{7})(7) + (\frac{1}{7})(5) + (\frac{4}{7})(2) \\ (\frac{4}{7})(7) + (\frac{2}{7})(5) + (\frac{1}{7})(2) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{31}{7} \\ \frac{27}{7} \\ \frac{40}{7} \end{bmatrix}$$

3. Transforming

(a) (3) Find the matrix of the transformation T with respect to the standard basis.

$$\begin{aligned} T[1,0,0] &= [0,-1,2]^T. \\ T[0,1,0] &= [2,0,-1]^T. \\ T[0,0,1] &= [-1,2,0]^T \end{aligned}$$

The matrix of the transformation T with respect to the standard basis is C.

(b) (4) Determine the range of the transformation $S(\vec{x}) = C\vec{x}$.
From above we know that C is invertible, therefore by the big theorem, the range is $\mathbb{R}^3$.

(c) (2) Determine if S is one-to-one. Determine if S is onto.
As above this is one-to-one and onto.

4. Alternate Spaces

For these problems use the space $\mathbb{P}^2$ with the inner product $< p(x), q(x) >= \int_0^1 p(x)q(x)\,dx$

(a) (4) Test if $p_1(x) = 1$ and $p_2(x) = x - 1/2$ are orthogonal.

$$\begin{aligned} \int_0^1 (1)(x-1/2)\,dx &= \\ \frac{1}{2}x^2 - \frac{1}{2}x\Big|_0^1 &= 0. \end{aligned}$$

They are orthogonal.

(b) (5) Generate an orthogonal basis for $\mathbb{P}^2$.

p_1 and p_2 are orthgonal, so only one more is needed. Start with $r_3(x) = x^2$.

$$\begin{aligned} < x^2, 1 > &= \int_0^1 x^2\,dx \\ &= \frac{1}{3}x^3\Big|_0^1 \\ &= \frac{1}{3}. \\ < 1, 1 > &= \int_0^1 1^2\,dx \\ &= x|_0^1 \\ &= 1. \\ < x^2, x-1/2 > &= \int_0^1 x^2(x-1/2)\,dx \\ &= \int_0^1 x^3 - \frac{1}{2}x^2\,dx \\ &= \frac{1}{4}x^4 - \frac{1}{6}x^3\Big|_0^1 \\ &= 1/12. \\ < x-1/2, x-1/2 > &= \int_0^1 (x-1/2)^2\,dx \\ &= \int_0^1 x^2 - x + 1/4\,dx \\ &= \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{4}x\Big|_0^1 \\ &= 1/12. \\ p_3(x) &= x^2 - \frac{1/12}{1/12}(x-1/2) - \frac{1/3}{1}1 \\ &= x^2 - x + 1/6. \end{aligned}$$

(c) (4) Generate an orthonormal basis from this basis.

The norms of p_1 and p_2 are calculated above.

$$\begin{aligned} < p_3, p_3 > &= \int_0^1 (x^2 - x + 1/6)^2\,dx \\ &= \int_0^1 x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{1}{3}x + \frac{1}{36}\,dx \\ &= \frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{4}{9}x^3 - \frac{1}{6}x^2 + \frac{1}{36}x\Big|_0^1 \\ &= \frac{1}{180}. \end{aligned}$$

$$
\begin{aligned}
o_1(x) &= 1. \\
o_2(x) &= \sqrt{12}(x - 1/2). \\
o_3(x) &= \sqrt{180}\left(x^2 - x + \frac{1}{6}\right).
\end{aligned}
$$

5. Thoughts

(a) (3) Suppose $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ is a dependent set. Use an example to show that $\vec{u}_1$ does not have to be dependent on all three other vectors.

If $\vec{u}_2 = [1,1,1]^T$, $\vec{u}_3 = [1,1,0]^T$, $\vec{u}_4 = [1,0,0]^T$, and $\vec{u}_1 = 2\vec{u}_2 + 3\vec{u}_3$ then the set is dependent, but $\vec{u}_4$ is not involved.

(b) (5) Prove that $\{p(x) : p(2) = 0\}$ is a subspace of $\mathbb{P}^2$.

Consider

$$\begin{aligned} ap_1(2) + bp_2(2) &= a(0) + b(0) \\ &= 0. \end{aligned}$$

Thus the set is closed and thus is a subspace.

(c) (3) Prove that $A^T A$ being invertible does not imply A is invertible.

As seen in linear regression, A need not be square and hence not invertible.