Math 314
Exam 1 Key

Instructions

1. Do NOT write your answers on these sheets. Nothing written on the test papers will be graded.
2. Please begin each section of questions on a new sheet of paper.
3. Do not write problems side by side.
4. Do not staple test papers.
5. Limited credit will be given for incomplete or incorrect justification.

Questions

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 7 & 8 & 3 \\ 3 & 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 7 & -5 \\ 4 & 29 & -21 \\ 2 & 13 & -9 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 7 \\ 20 \\ 64 \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} 8 \\ 7 \\ 51 \end{bmatrix}.$$

1. Matrix A (6 each)

   (a) Calculate A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 7 & 8 & 3 & 0 & 1 & 0 \\ 3 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \sim \quad \begin{array}{l} R_2 \leftarrow -7R_1 + R_2 \\ R_3 \leftarrow -3R_1 + R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -4 & -7 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right] \sim \quad R_1 \leftarrow -1R_2 + R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 8 & -1 & 0 \\ 0 & 1 & -4 & -7 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right] \sim \quad \begin{array}{l} R_1 \leftarrow -5R_3 + R_1 \\ R_2 \leftarrow 4R_3 + R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 23 & -1 & -5 \\ 0 & 1 & 0 & -19 & 1 & 4 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$$

   (b) Find all solutions to $A\vec{x} = \vec{0}$.
   Because the matrix is invertible, only the trivial solution exists.

$$\vec{x} = [0, 0, 0]^T.$$

   (c) Find all solutions to $A\vec{x} = \vec{c}$.

$$\begin{bmatrix} 23 & -1 & -5 \\ -19 & 1 & 4 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 7 \\ 51 \end{bmatrix} = \begin{bmatrix} -78 \\ 59 \\ 27 \end{bmatrix}$$

   (d) Calculate $\det(A)$.

$$\det(A) = 1.$$

2. Matrix B (6 each)

(a) Find all solutions to $B\vec{x} = \vec{c}$.

$$\left[\begin{array}{rrr|r} 1 & 7 & -5 & 8 \\ 4 & 29 & -21 & 7 \\ 2 & 13 & -9 & 51 \end{array}\right] \sim \quad \begin{array}{l} R_2 \leftarrow -4R_1 + R_2 \\ R_3 \leftarrow -2R_1 + R_3 \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 7 & -5 & 8 \\ 0 & 1 & -1 & -25 \\ 0 & -1 & 1 & 35 \end{array}\right] \sim \quad R_3 \leftarrow 1R_2 + R_3$$

$$R_1 \leftarrow -7R_2 + R_1$$

$$\left[\begin{array}{rrr|r} 1 & 7 & -5 & 8 \\ 0 & 1 & -1 & -25 \\ 0 & 0 & 0 & 10 \end{array}\right] \sim$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 2 & 183 \\ 0 & 1 & -1 & -25 \\ 0 & 0 & 0 & 10 \end{array}\right]$$

There are no solutions.

(b) Find all solutions to $B\vec{x} = \vec{0}$.
Based on above calculations

$$[-2, 1, 1]^T x_3$$

(c) Calculate $\det(B)$.

$$\det(B) = 0.$$

(d) Is $T(\vec{x}) = B\vec{x}$ one-to-one? onto?
T is not one-to-one, because many vectors map to $\vec{0}$. By the Big Theorem it can also not be onto.

3. Transformations (6 each)

(a) Find the matrix of the linear transformation T that maps
$T([1,1,0]^T) = [2,15,6]^T$, and $T([1,0,1]^T) = [2,10,7]^T$, and $T([0,1,1]^T) = [2,11,7]^T$.

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 15 & 10 & 11 \\ 6 & 7 & 7 \end{bmatrix}.$$

To make this easier to solve not that if $XA = B$ then $(XA)^T = B^T$ or $A^T X^T = B^T$ so we can solve this using the transpose.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & 15 & 6 \\ 1 & 0 & 1 & 2 & 10 & 7 \\ 0 & 1 & 1 & 2 & 11 & 7 \end{array}\right] \sim \quad R_2 \leftarrow -1R_1 + R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & 15 & 6 \\ 0 & -1 & 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 2 & 11 & 7 \end{array}\right] \sim \quad R_3 \leftarrow 1R_2 + R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & 15 & 6 \\ 0 & -1 & 1 & 0 & -5 & 1 \\ 0 & 0 & 2 & 2 & 6 & 8 \end{array}\right] \sim \quad \begin{array}{l} R_1 \leftarrow 1R_2 + R_1 \\ R_2 \leftarrow -R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 10 & 7 \\ 0 & 1 & -1 & 0 & 5 & -1 \\ 0 & 0 & 2 & 2 & 6 & 8 \end{array}\right] \sim \quad \begin{array}{l} R_1 \leftarrow -\frac{1}{2}R_3 + R_1 \\ R_2 \leftarrow \frac{1}{2}R_3 + R_2 \\ R_3 \leftarrow \frac{1}{2}R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 7 & 3 \\ 0 & 1 & 0 & 1 & 8 & 3 \\ 0 & 0 & 1 & 1 & 3 & 4 \end{array}\right]$$

The matrix is

$$\begin{bmatrix} 1 & 1 & 1 \\ 7 & 8 & 3 \\ 3 & 3 & 4 \end{bmatrix}$$

(b) Determine if the linear transformation S that maps
$S([1,0,0]^T) = [4,1,2]^T$, $S([0,1,0]^T) = [2,1,1]^T$, $S([0,0,1]^T) = [4,1,2]^T$
is one-to-one and if it is onto.

Because $[1,0,0]^T$ and $[0,0,1]^T$ are mapped to the same vector, this is not one-to-one. By the Big Theorem, this can also not be onto.

4. Connections (4 each)

(a) Prove that for two matrices A and B if AB is invertible, then A necessarily has a right inverse. Because AB is invertible there exists E such that $(AB)E = I$. Thus

$$\begin{aligned} (AB)E &= I. \\ A(BE) &= I. \end{aligned}$$

Thus by associativity of matrix multiplication the right inverse of A is BE.

(b) Prove that if $\vec{w}$ is in the span of $\vec{u}$ and $\vec{v}$ that $\{\vec{u}, \vec{v}, \vec{w}\}$ is a dependent set. Do not use the Big Theorem. Because $\vec{w}$ is in the span

$$\begin{aligned} \vec{w} &= a\vec{u} + b\vec{v}. \\ \vec{0} &= a\vec{u} + b\vec{v} + \vec{w}. \end{aligned}$$

Thus $[a, b, 1]^T$ is a non-trivial solution to $A\vec{x} = \vec{0}$.