Partial Fraction Decomposition

The algebraic manipulation partial fraction decomposition works because the component fractions form a basis for the subspace of rational functions that include the expression being manipulated. Aspects of this are demonstrated through the sections below.

1. Partial Fraction Decomposition: Basis

$$\frac{2x^2+7x-32}{(x+3)(x-2)(x-4)} = \frac{A}{x+3} + \frac{B}{x-2} + \frac{C}{x-4}.$$

   (a) Independence
   When set up correctly the component rational expressions in partial fraction decomposition are independent.
      i. Setup the homogeneous equation for $\frac{1}{x+3}$, $\frac{1}{x-2}$, and $\frac{1}{x-4}$ to test for linear independence.
      ii. How does this look similar to the setup for partial fraction decomposition? How is it different?
      iii. Clear the denominators in this equation (same process as in partial fraction decomposition).
      iv. Collect the terms.
      v. Setup the system of equations this produces.
      vi. Row reduce to demonstrate that these vectors (rational functions) are linearly independent.

   (b) Span
   When set up correctly the rational function being decomposed is in the span of the component rational expressions.
      i. Setup the non-homogeneous equation to test if $\frac{2x^2+7x-32}{(x+3)(x-2)(x-4)}$ is in the span of the three component ration expressions.
      ii. Compare this to the setup for partial fraction decomposition.
      iii. Clear the denominators in this equation (same process as in partial fraction decomposition).
      iv. Collect the terms.
      v. Setup the system of equations this produces.
      vi. Row reduce to demonstrate that this vector (rational function) is in the span of the other vectors (rational functions).

2. Irreducible quadratics

   (a) Irreducible quadratics introduce two vectors.

$$\frac{5x^2-10x+11}{(x-3)(x^2+4)}$$

      i. Show that the rational expression is not in the span of

$$A\left(\frac{1}{x-3}\right) + B\left(\frac{1}{x^2+4}\right).$$

      ii. Show that the rational expression is in the span of

$$A\left(\frac{1}{x-3}\right) + B\left(\frac{x}{x^2+4}\right) + C\left(\frac{1}{x^2+4}\right).$$

      iii. How does this latter relate to the usual setup?

$$\frac{A}{x-3} + \frac{Bx+C}{x^2+4}$$

(b) When setup incorrectly partial fraction decomposition produces useless duplication.

$$\frac{x+4}{x^2-4} = \frac{A}{x-2} + \frac{Bx+C}{x^2-4}.$$

i. Show that the three vectors (rational expressions) are not linearly independent.
ii. Why would we want to avoid dependent vectors? Thinking about bases may help.

3. Types of Errors
Not all setup errors in partial fraction decomposition produce inaccurate results.

$$\frac{3x^2-5x-7}{(x-3)(x^2-4)}$$

(a) Correct setup: $\frac{3x^2-5x-7}{(x-3)(x^2-4)} = \frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{x+2}$.
i. Complete the partial fraction decomposition process for this rational function.
ii. How does your work show that the three component vectors (rational functions) are independent?
iii. How does your work show that the initial vector (rational function) is in the span of the three component vectors (rational functions)?

(b) Sloppy setup: $\frac{3x^2-5x-7}{(x-3)(x^2-4)} = \frac{A}{x-3} + \frac{Bx+C}{x^2-4}$.
i. Show that the three vectors (rational functions) on the right are linearly independent.
ii. Show that the vector (rational function) on the left is in the span of the three, component vectors (rational functions) on the right.
iii. Solve for the coefficients and check that the equality holds by adding the rational functions.

(c) What is the connection between both setups working and bases?

(d) Note that complex roots can be used for partial fraction decomposition. How could the problem in 2a be setup differently using complex roots?

(e) Why would the partial fraction decomposition algorithm demand quadratic functions be factored fully?