Theorem's We Know

1. Basic Theorems

   (a) There exists a first parallel.

   (b) The angle of parallelism is the same on the left and right.

   (c) The angle of parallelism is less than a right angle.

2. Omega Triangles

   (a) (Pasch's Axiom Hyperbolic) If a line k contains a point of the interior of an omega triangle and a vertex, it must also cross the side opposite the vertex.

   (b) (Exterior Angle Theorem Hyperbolic) The measure of an exterior angle of an omega triangle is greater than the measure of the opposite interior angle.

3. Omega Triangle Congruence

   (a) (Angle Side) If the finite sides and one pair of angles of two omega triangles are congruent, the omega triangles are congruent.

   (b) (Angle Angle) If corresponding angles of omega triangles are congruent, the omega triangles are congruent.

4. Quadrilaterals

   (a) The summit angles of a Saccheri quadrilateral are congruent.

   (b) The summit angles of a Saccheri quadrilateral are acute.

   (c) No rectangle exists.

   (d) The angle sum of a triangle is less than π.

   (e) The fourth angle of a Lambert quadrilateral is acute.

   (f) The sides of a Lambert quadrilateral adjacent to the acute angle are longer than the opposing sides.

Chapter 1

Omega Triangles

Theorem 1 (Pasch's Axiom Hyperbolic) *If a line k contains a point of the interior of an omega triangle and a vertex, it must also cross the side opposite the vertex.*

Proof: Let $\overline{A\Omega}$ and $\overline{B\Omega}$ be sensed parallels. WoLOG let k be a line through A and a point C in the interior. Note $m\angle BAC < m\angle BA\Omega$ by construction. Because $\angle BA\Omega$ is the angle of parallelism, line k must intersect ray $\overline{B\Omega}$.
Suppose line k contains C in the interior and Ω. Suppose $\overline{AC}$ intersects $\overline{B\Omega}$ at D. By Pasch's axiom (normal) $\overline{C\Omega}$ must intersect either $\overline{AB}$ or $\overline{BD}$. If k intersects $\overline{BD}$ then there would exist two sensed parallels in the same direction to $\overline{A\Omega}$ at that point of intersection. Because this is impossible k intersects $\overline{AB}$.
□

Theorem 2 (Exterior Angle Theorem Hyperbolic) *The measure of an exterior angle of an omega triangle is greater than the measure of the opposite interior angle.*

Proof: Let $AB\Omega$ be an omega triangle. Let C be a point such that $C - A - B$. We show that $m\angle\Omega AC > m\angle\Omega BA$. This proof is by contradiction.
Case 1: Suppose $m\angle CA\Omega < m\angle AB\Omega$. Let Z be a point in the interior such that $\angle ABZ \cong \angle CA\Omega$. Because this is less than the angle of parallelism, $\overline{BZ} \cap \overline{A\Omega} = D$. Now $\angle CA\Omega$ is the exterior angle of a normal $\triangle ABD$ with an equal, opposite interior angle. This is not possible.
Case 2: Suppose $\angle CA\Omega \cong \angle AB\Omega$. Let E be the midpoint of $\overline{AB}$. Let D be the foot of the perpendicular from E to $\overline{A\Omega}$. Let F be the point on $\overline{B\Omega}$ such that $|FB| = |AD|$ and F and D are on opposite sides of $\overline{AB}$. Note $\angle FBE \cong \angle DAE$ because they are supplementary to $\angle AB\Omega$ and $\angle CA\Omega$ respectively. By SAS $\triangle FBE \cong \triangle DAE$. Thus $\angle BEF \cong \angle AED$ and $F - E - D$. Because $\angle ADE$ is right, so is $\angle BFE$. Thus the angle of parallelism at F to $\overline{A\Omega}$ is a right angle. This is a contradiction to the angle of parallelism being less than a right angle. Thus $m\angle\Omega AC > m\angle\Omega BA$. □

Chapter 2

Omega Triangle Congruence

Theorem 3 (Angle Side) *If the finite sides and one pair of angles of two omega triangles are congruent, the omega triangles are congruent.*

Proof: This is a proof by contradiction. Consider omega triangles $AB\Omega$ and $CD\Lambda$ with $\overline{AB} \cong \overline{CD}$ and $\angle AB\Omega \cong CD\Lambda$. Suppose $\angle BA\Omega \ncong \angle DC\Lambda$. WoLOG $m\angle BA\Omega > m\angle DC\Lambda$. Let P be an interior point of $BA\Omega$ such that $\angle BAP \cong \angle DC\Lambda$. By previous theorem it must intersect $\overline{B\Omega}$ at a point E. Let F be a point on $\overline{D\Lambda}$ such that $\overline{DF} \cong \overline{BE}$. Note $\triangle BEA \cong \triangle DFC$ by SAS. However, this implies $\angle DCF \cong \angle BAE \cong \angle DC\Lambda$. Further $\overline{C\Lambda} \cap \overline{D\Lambda} \neq \emptyset$. Thus the second angle must be congruent. $\square$

Theorem 4 (Angle Angle) *If corresponding angles of omega triangles are congruent, the omega triangles are congruent.*

Proof: Let $AB\Omega$ and $CD\Lambda$ be omega triangles with $\angle BA\Omega \cong \angle DC\Lambda$ and $\angle AB\Omega \cong \angle CD\Lambda$. This is a proof by contradiction. WoLOG suppose $|CD| > |AB|$. Let $\overline{E\Lambda}$ be the sensed parallel to $\overline{C\Lambda}$ through E. By the previous theorem $AB\Omega \cong CE\Lambda$. Thus $\angle CE\Lambda \cong \angle AB\Omega \cong \angle CD\Lambda$. Now, $\overline{D\Lambda}$ and $\overline{E\Lambda}$ are sensed parallels to $\overline{C\Lambda}$, and are therefore sensed parallels to each other. Thus $ED\Lambda$ is an omega triangle. However we have exterior $\angle CE\Lambda \cong \angle ED\Lambda$ which contradicts a previous theorem. Thus $|DC| = |AB|$. $\square$

Chapter 3

Quadrilaterals

Theorem 5 *The summit angles of a Saccheri quadrilateral are congruent.*

Proof: Let $ABCD$ be a Saccheri quadrilateral with base $\overline{AB}$ and summit $\overline{DC}$. Thus $\overline{AD} \cong \overline{BC}$, and $m\angle DAB = m\angle CBA = \pi/2$. Construct diagonals $\overline{DB}$ and $\overline{CA}$. Note $\overline{AB} \cong \overline{AB}$. Thus $\triangle DAB \cong \triangle CBA$ by SAS. Now $\overline{DB} \cong \overline{CA}$ being corresponding parts of congruent triangles. Note $\overline{DC} \cong \overline{DC}$. Thus $\triangle ADC \cong \triangle BCD$. Hence $\angle ADC \cong \angle BCD$ being corresponding parts of congruent triangles. $\square$

Theorem 6 *The summit angles of a Saccheri quadrilateral are acute.*

Proof: Let $ABCD$ be a Saccheri quadrilateral with base $\overline{AB}$ and summit $\overline{DC}$. Thus $\overline{AD} \cong \overline{BC}$, and $m\angle DAB = m\angle CBA = \pi/2$. Construct left-sensed parallels $\overline{C\Omega}$ and $\overline{D\Omega}$ to $\overline{AB}$. Let E be a point such that $E - D - C$. By previous theorem $m\angle ED\Omega > m\angle DC\Omega$. Also $\angle AD\Omega \cong \angle BC\Omega$ by an omega triangle congruency theorem. Thus $m\angle EDA > m\angle DCB$, and $m\angle EDA > m\angle CDA$. Also $m\angle EDA + m\angle CDA = \pi$, so $m\angle CDA < \pi/2$. $\square$

Theorem 7 *No rectangle exists.*

Proof: This is a proof by contradiction. Suppose $ABCD$ is a rectangle. Note $ABCD$ is a Saccheri quadrilateral. By the previous theorem the summit angles are acute. This contradicts that the summit angles are right angles per definition of rectangle. Thus no rectangle exists. $\square$

Theorem 8 *The fourth angle of a Lambert quadrilateral is acute.*

Proof:
Case 1: Suppose the fourth angle is obtuse. This results in a quadrilateral with angle sum greater than 2π. Thus case 1 is not possible.
Case 2: Suppose the fourth angle is a right angle. This results in a rectangle. Thus case 2 is not possible.
Case 3: The fourth angle is acute. $\square$

Theorem 9 *The sides of a Lambert quadrilateral adjacent to the acute angle are longer than the opposing sides.*

Proof: Let $ABCD$ be a Lambert quadrilateral with acute $\angle CDA$.
Case 1: Suppose $|AD| < |BC|$. Let E be the point such that $B - E - C$ and $|BE| = |AD|$. Note $ABED$ is a Saccheri quadrilateral. Thus $\angle BED$ is acute, and $\angle DEC$ must be obtuse. However this implies the angle sum of $\triangle DEC$ is greater than the sum of a right angle and an obtuse angle. Thus case 1 is not possible.
Case 2: Suppose $|AD| = |BC|$. Thus $DABC$ is a Saccheri quadrilateral with summit angles $\angle BCD$ and $\angle ADC$. However $\angle BCD$ is a right angle. Thus $|AD| \neq |BC|$. Thus case 2 is not possible.
Case 3: $|AD| > |BC|$. $\square$