Math 231 Introduction to Discrete Mathematics
Final Exam Key

Instructions

1. Do NOT write your answers on these sheets. Nothing written on the test papers will be graded.
2. Please begin each section of questions on a new sheet of paper.
3. Please do not write answers side by side.
4. Please do not staple your test papers together.
5. Limited credit will be given for incomplete or incorrect justification.

Questions

1. Enumeration (4 each)

   (a) If Guido selects two scoops of ice cream that can be seen above the top of a cone, and he chooses to not let the cones look exactly like a previous cone until he has tried all options, how many times must he order ice cream if there are 10 different varieties?

   Because the ice cream can be seen, order matters. He is not restricted to different types.

   $$\boxed{10}\boxed{10} = 10^2$$

   (b) Guido has five (5) types of cookies which he is arranging on a long tray. If he can fit 20 cookies next to each other on the tray, and he will not put two of the same sort of cookie in a row, how many ways can he arrange a tray? Note he has at least 20 of each type of cookie.

   Order matters, and the objects are distinguishable. Except for the first position there are only four choices: the four not used in the previous position.

   $$\boxed{5}\boxed{4}\boxed{\ldots}\boxed{4} = 5 \cdot 4^{19}$$

   Note, the trays are not actually different in pattern if looked at from left or right, so the count is half the above.

   (c) Guido is arranging two types of truffles on another tray. If he will put two of the same but not three of the same in a row, how many ways can he arrange 5 truffles in a row? He has more than five of each type.

   There are 2^5 arrangements of the truffles. Of these 2 ways use 5 of the same. If four are the same there are two ways to choose which of the two is four and two ways to arrange these (e.g., 14, 41) for 4 options. For three of the same the options are: aabbb, aaabb, abbba, babbb, bbbab (each two ways). So the total count is

   $$2^5 - (2 + 4 + 10) = 16.$$

   Alternately consider that for 3 of the same, only 3 choices are made times 3 ways to select which is the triple with two options for each choice for $3 \cdot 2^3$. For four of the same, only 2 choices are made times 2 ways to select which is the quad with two options for each for $2 \cdot 2^2$. For five of the same, only 1 position is selected with two options for each for 2^1. However the above overlap with the count of 3 in a row double counting 4 in a row and triple counting 5 in a row, so the count is

   $$2^5 - 3 \cdot 2^3 + 2 \cdot 2^2 = 32 - (24 - 8) = 16.$$

(d) If passwords have 30 characters which may be upper or lower case letters and also numbers, and the password checking software forces waiting 10 minutes after 3 failed attempts, how long might a password cracking system need to find the correct password by trying all options?

62	62	...	62

Total number of passwords: $(26 + 26 + 10)^{30} = 62^{30}$.
Sets of three: $\frac{62^{30}}{3}$
Time: $\frac{62^{30}}{3} \cdot 10$

(e) If by snooping, you know that the letters of the password are 'qsccfttgbbhuuj' how many passwords must be tested?

14	13	...	1

Note there are four letters that appear twice (indistinguishable). Total arrangements of these are

$$\frac{14!}{2!2!2!2!}$$

(f) Write a recurrence relation for the number of strings consisting of the characters 0,1,2 with no consecutive 0's.

Note $a_1 = 3$ and $a_2 = 3^2 - 1 = 8$ (all strings of length 2 except 00). For every string of length $n - 1$ the characters 1 and 2 can be appended. For strings of length $n - 1$ not ending in 0, the character 0 can be appended. Because these strings of length $n - 1$ ended in 1 or 2 they were produced by appending this character to some (any) string of length $n - 2$.

$$a_n = 2a_{n-1} + 2a_{n-2}.$$

2. Graph Theory

   (a) Answer the following for the graph in Figure 1. (4 each)

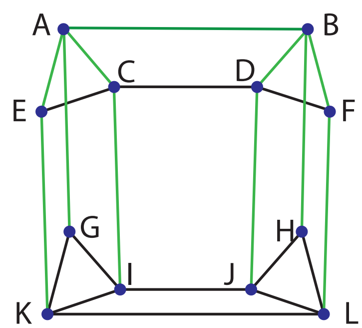
      i. Find a maximum clique.
      A,B,C or any other K_3 is a maximum clique.

      ii. Find a maximum vertex independence set.
      A,F,J,K (one from each K_3).

      iii. Find a maximum edge independence set.
      AB, EC, DF, GI, JH, KL

      iv. Give the min and max degree.
      $\delta(G_1) = 3$
      $\Delta(G_1) = 4$

      v. Find a spanning tree.



   (b) (4 each) Determine which of the graphs in Figures 1 to 3 are isomorphic. Provide an isomorphism for isomorphic pairs and a reason why the others are not.

   Graphs 1 and 2 are isomorphic.

G_1	A	B	C	D	E	F	G	H	I	J	K	L
G_2	A	J	I	B	E	F	G	H	C	L	K	D

   Graphs 1 and 3 (hence 2 and 3) are not isomorphic. Note that in graph 3, two vertices of degree 3 (e.g., A, I) are adjacent, while in graphs 1 and 2 no such pairs of degree 3 vertices exist.

3. Relations

(a) Equivalence Relation

The relation is on $X = \{(a,b,c) : a,b,c \in \mathbb{R}\}$ with $(a_1,b_1,c_1)R(a_2,b_2,c_2)$ if and only if $(a_1,b_1,c_1) = k(a_2,b_2,c_2)$ for some $k \in \mathbb{R} - \{0\}$.

i. (5) Prove this is an equivalence relation.

Reflexive: Consider $(a,b,c) \in X$. Note that $(a,b,c) = 1(a,b,c)$. Thus every element is related to itself (the relation is reflexive).

Symmetric: Consider $(a_1,b_1,c_1),(a_2,b_2,c_2) \in X$ such that $(a_1,b_1,c_1)R(a_2,b_2,c_2)$. By definition of the relation

$$\begin{aligned} (a_1,b_1,c_1) &= k(a_2,b_2,c_2). \\ \frac{1}{k}(a_1,b_1,c_1) &= (a_2,b_2,c_2). \end{aligned}$$

Because $1/k \in \mathbb{R}$ $(a_2,b_2,c_2)R(a_1,b_1,c_1)$, and the relation is symmetric.

Transitive: Consider $(a_1,b_1,c_1),(a_2,b_2,c_2),(a_3,b_3,c_3) \in X$ such that $(a_1,b_1,c_1)R(a_2,b_2,c_2)$ and $(a_2,b_2,c_2)R(a_3,b_3,c_3)$. By definition of the relation

$$\begin{aligned} (a_1,b_1,c_1) &= j(a_2,b_2,c_2). \\ (a_2,b_2,c_2) &= k(a_3,b_3,c_3). \\ (a_1,b_1,c_1) &= jk(a_3,b_3,c_3). \end{aligned}$$

Because $jk \in \mathbb{R}$ $(a_1,b_1,c_1)R(a_3,b_3,c_3)$ and the relation is transitive.

Because the relation is reflexive, symmetric, and transitive it is an equivalence relation.

ii. (4) Write at least three elements of the equivalence classes $[(1,1,1)]$ and $[(1,0,3)]$

$$[(1,1,1)] = \{(1,1,1),(-1,-1,-1),(27,27,27),\ldots\}$$
$$[(1,0,3)] = \{(1,0,3),(-1,0,-3),(14,0,42),\ldots\}$$

iii. (2) Do all the equivalence classes in this relation have the same number of elements?

No. Note $[(0,0,0)] = \{(0,0,0)\}$. The others are infinite.

(b) Poset

The relation is on $X = \{(a,b,c) : a,b,c \in \mathbb{Z}^+ \cup \{0\}\}$ with $(a_1,b_1,c_1)R(a_2,b_2,c_2)$ if and only if $2^{a_1}3^{b_1}5^{c_1} \le 2^{a_2}3^{b_2}5^{c_2}$.

i. (5) Prove this is a partial ordering.

Reflexive: Consider $(a,b,c) \in X$. Note $2^a3^b5^c \le 2^a3^b5^c$ by definition of $\le$ (equals). Thus the relation is reflexive.

Anti-symmetric: Consider $(a_1,b_1,c_1),(a_2,b_2,c_2) \in X$ such that $(a_1,b_1,c_1)R(a_2,b_2,c_2)$ and $(a_2,b_2,c_2)R(a_1,b_1,c_1)$. By definition of the relation

$$\begin{aligned} 2^{a_1}3^{b_1}5^{c_1} &\le 2^{a_2}3^{b_2}5^{c_2}, \\ 2^{a_2}3^{b_2}5^{c_2} &\le 2^{a_1}3^{b_1}5^{c_1}. \\ 2^{a_1}3^{b_1}5^{c_1} &= 2^{a_2}3^{b_2}5^{c_2}. \\ a_1 &= a_2. \\ b_1 &= b_2. \\ c_1 &= c_2. \end{aligned}$$

Transitive: Consider $(a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3) \in X$ such that $(a_1, b_1, c_1)R(a_2, b_2, c_2)$ and $(a_2, b_2, c_2)R(a_3, b_3, c_3)$. By definition of the relation

$$\begin{aligned} 2^{a_1}3^{b_1}5^{c_1} &\leq 2^{a_2}3^{b_2}5^{c_2}, \\ 2^{a_2}3^{b_2}5^{c_2} &\leq 2^{a_3}3^{b_3}5^{c_3}. \\ 2^{a_1}3^{b_1}5^{c_1} &\leq 2^{a_3}3^{b_3}5^{c_3}. \end{aligned}$$

The latter is by transitivity of $\leq$. Thus the relation is transitive.
Because the relation is reflexive, anti-symmetric and transitive it is a poset.

ii. (4) Write two comparable and two incomparable items–if they exist.
$(1, 2, 3)$ and $(4, 5, 6)$ are comparable.
No pairs are incomparable. Every pair of integers has a lesser integer.

iii. (4) Find the lub and glb of $(5, 0, 1)$ and $(1, 1, 2)$.

$$\begin{aligned} 2^5 3^0 5^1 &= 160. \\ 2^1 3^1 5^2 &= 50. \end{aligned}$$

Thus the lub is $(5, 0, 1)$ and the glb is $(1, 1, 2)$.

iv. (4) List a minimal and a maximal element if they exist.
The minimal element is $(0, 0, 0)$ because $2^0 3^0 5^0 = 1$ which is the smallest non-zero, non-negative integer. There is no maximal element, because there is always a bigger integer.

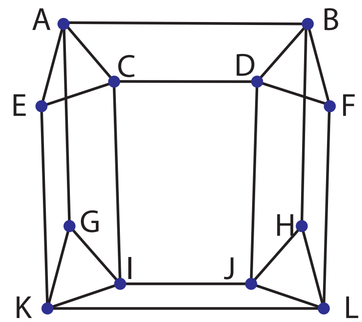


Figure 1: Graph 1

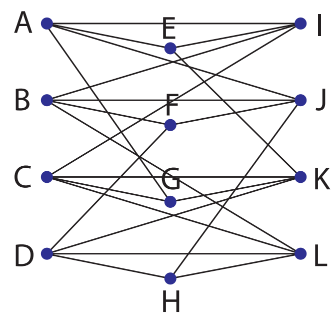


Figure 2: Graph 2

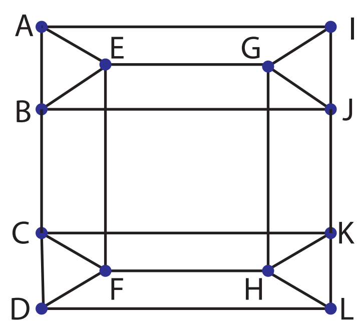


Figure 3: Graph 3