Derivation of the Inverse Hyperbolic Trig Functions

$$y = \sinh^{-1} x.$$

By definition of an inverse function, we want a function that satisfies the condition

$$\begin{aligned}
x &= \sinh y \\
&= \frac{e^y - e^{-y}}{2} \text{ by definition of } \sinh y \\
&= \frac{e^y - e^{-y}}{2} \left(\frac{e^y}{e^y}\right) \\
&= \frac{e^{2y} - 1}{2e^y}. \\
2e^y x &= e^{2y} - 1. \\
e^{2y} - 2xe^y - 1 &= 0. \\
(e^y)^2 - 2x(e^y) - 1 &= 0. \\
e^y &= \frac{2x + \sqrt{4x^2 + 4}}{2} \\
&= x + \sqrt{x^2 + 1}. \\
\ln(e^y) &= \ln(x + \sqrt{x^2 + 1}). \\
y &= \ln(x + \sqrt{x^2 + 1}).
\end{aligned}$$

Thus

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}).$$

Next we compute the derivative of $f(x) = \sinh^{-1} x$.

$$\begin{aligned}
f'(x) &= \frac{1}{x + \sqrt{x^2 + 1}} \left(1 + \frac{1}{2}(x^2 + 1)^{-1/2}(2x)\right) \\
&= \frac{1}{\sqrt{x^2 + 1}}.
\end{aligned}$$

$$y = \cosh^{-1} x.$$

By definition of an inverse function, we want a function that satisfies the condition

$$\begin{aligned}
x &= \cosh y \\
&= \frac{e^y + e^{-y}}{2} \text{ by definition of } \cosh y \\
&= \frac{e^y + e^{-y}}{2} \left(\frac{e^y}{e^y} \right) \\
&= \frac{e^{2y} + 1}{2e^y}. \\
2e^y x &= e^{2y} + 1. \\
e^{2y} - 2xe^y + 1 &= 0. \\
(e^y)^2 - 2x(e^y) + 1 &= 0. \\
e^y &= \frac{2x + \sqrt{4x^2 - 4}}{2} \\
&= x + \sqrt{x^2 - 1}. \\
\ln(e^y) &= \ln(x + \sqrt{x^2 - 1}). \\
y &= \ln(x + \sqrt{x^2 - 1}).
\end{aligned}$$

Thus

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}).$$

Next we compute the derivative of $f(x) = \cosh^{-1} x$.

$$\begin{aligned}
f'(x) &= \frac{1}{x + \sqrt{x^2 - 1}} \left(1 + \frac{1}{2}(x^2 - 1)^{-1/2}(2x) \right) \\
&= \frac{1}{\sqrt{x^2 - 1}}.
\end{aligned}$$

$$y = \tanh^{-1} x.$$

By definition of an inverse function, we want a function that satisfies the condition

$$\begin{aligned}
x &= \tanh y \\
&= \frac{e^y - e^{-y}}{e^y + e^{-y}} \text{ by definition of } \tanh y \\
&= \frac{e^y - e^{-y}}{e^y + e^{-y}} \left(\frac{e^y}{e^y}\right) \\
&= \frac{e^{2y} - 1}{e^{2y} + 1}. \\
x(e^{2y} + 1) &= e^{2y} - 1. \\
(x-1)e^{2y} + (x+1) &= 0. \\
e^{2y} &= -\frac{x+1}{x-1}. \\
\ln(e^{2y}) &= \ln\left(-\frac{x+1}{x-1}\right). \\
2y &= \ln\left(-\frac{x+1}{x-1}\right). \\
y &= \frac{1}{2}\ln\left(-\frac{x+1}{x-1}\right) \\
&= \frac{1}{2}(\ln(x+1) - \ln(-[x-1])) \\
&= \frac{1}{2}(\ln(x+1) - \ln(1-x)).
\end{aligned}$$

Thus

$$\tanh^{-1} x = \frac{1}{2}(\ln(x+1) - \ln(1-x)).$$

Next we compute the derivative of $f(x) = \tanh^{-1} x$.

$$\begin{aligned}
f'(x) &= \frac{1}{2}\left(\frac{1}{x+1} - \frac{1}{1-x}(-1)\right) \\
&= \frac{1}{2}\left(\frac{1}{x+1} + \frac{1}{1-x}\right) \\
&= \frac{1}{1-x^2}.
\end{aligned}$$

$$y = \text{sech}^{-1} x.$$

By definition of an inverse function, we want a function that satisfies the condition

$$\begin{aligned}
x &= \text{sech} y \\
&= \frac{2}{e^y + e^{-y}} \text{ by definition of sech} y \\
&= \frac{2}{e^y + e^{-y}} \left(\frac{e^y}{e^y} \right) \\
&= \frac{2e^y}{e^{2y} + 1}. \\
x(e^{2y} + 1) &= 2e^y. \\
xe^{2y} - 2e^y + x &= 0. \\
e^y &= \frac{-(-2) + \sqrt{(-2)^2 - 4(x)(x)}}{2x} \\
&= \frac{2 + \sqrt{4(1 - x^2)}}{2x} \\
&= \frac{2 + 2\sqrt{1 - x^2}}{2x} \\
&= \frac{1 + \sqrt{1 - x^2}}{x}. \\
y &= \ln \left(\frac{1 + \sqrt{1 - x^2}}{x} \right) \\
&= \ln(1 + \sqrt{1 - x^2}) - \ln x.
\end{aligned}$$

Thus

$$\text{sech}^{-1} x = \ln(1 + \sqrt{1 - x^2}) - \ln x.$$

Next we compute the derivative of $f(x) = \text{sech}^{-1} x$.

$$\begin{aligned}
f'(x) &= \frac{1}{1 + \sqrt{1 - x^2}} \left(\frac{1}{2}(1 - x^2)^{-1/2}(-2x) \right) - \frac{1}{x} \\
&= -\frac{1}{x\sqrt{1 - x^2}}.
\end{aligned}$$