Deriving Inverse Functions

Derive a formula for the inverse hyperbolic tangent. Find and simplify the derivative of the inverse hyperbolic tangent. The process will be similar to the derivation of the inverse hyperbolic sine shown below.

$$y = \sinh^{-1} x.$$

By definition of an inverse function, we want a function that satisfies the condition

$$\begin{aligned}
x &= \sinh y \\
&= \frac{e^y - e^{-y}}{2} \text{ by definition of } \sinh y \\
&= \frac{e^y - e^{-y}}{2}\left(\frac{e^y}{e^y}\right) \\
&= \frac{e^{2y} - 1}{2e^y}. \\
2e^y x &= e^{2y} - 1. \\
e^{2y} - 2xe^y - 1 &= 0. \\
(e^y)^2 - 2x(e^y) - 1 &= 0. \\
e^y &= \frac{2x + \sqrt{4x^2 + 4}}{2} \\
&= x + \sqrt{x^2 + 1}. \\
\ln(e^y) &= \ln(x + \sqrt{x^2 + 1}). \\
y &= \ln(x + \sqrt{x^2 + 1}).
\end{aligned}$$

Thus

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}).$$

Next we compute the derivative of $f(x) = \sinh^{-1} x$.

$$\begin{aligned}
f'(x) &= \frac{1}{x + \sqrt{x^2 + 1}}\left(1 + \frac{1}{2}(x^2 + 1)^{-1/2}(2x)\right) \\
&= \frac{1}{\sqrt{x^2 + 1}}.
\end{aligned}$$