Assignment 6

1. Generating Taylor Polynomials
   (a) Find the first nine derivatives of \( f(x) = \tan(x) \).
   (b) Evaluate the first nine derivatives of \( \tan(x) \) at \( x = 0 \).
   (c) Write the ninth degree Taylor polynomial about \( x = 0 \) for \( \tan(x) \).
   (d) Generate the Taylor polynomial using Mathematica’s built in routine.

2. Accuracy of Taylor Polynomials
   (a) Generate the third, fifth, and seventh degree Taylor polynomials for \( \sin(x) \) about \( c = 0 \).
   (b) Graph the above approximations along with \( \sin(x) \) on the interval \([-3\pi/2, 3\pi/2]\).
   (c) Graph the above approximations along with \( \sin(x) \) using the window \{\{Pi/2–0.1, Pi/2+0.1\}, \{0.995, 1.005\}\}.
   (d) Generate the third, fifth, and seventh degree Taylor polynomials for \( \sin(x) \) about \( c = \pi/2 \).
   (e) Graph the above approximations along with \( \sin(x) \) on the interval \([0, \pi]\).
   (f) Graph the above approximations along with \( \sin(x) \) using the window \{\{2.9, 3.1\}, \{0, 0.4\}\}.
   (g) What happens to the error as \( x \) moves away from \( c \)?

3. Inner Product Approximation
   \[ L_0(x) = 1, \quad L_1(x) = x, \quad L_2(x) = x^2 - \frac{\pi^2}{3}, \quad L_3(x) = x^3 - \frac{3\pi^2}{5}x. \]
   (a) Calculate \( \int_{-\pi}^{\pi} |T_3(x) - \sin(x)| \, dx \) where \( T_3(x) \) is the 3rd degree Taylor Polynomial for \( \sin(x) \) about \( c = 0 \).
      Note that this gives an estimate of the total error of the estimate on this interval.
   (b) Calculate the following integrals.

   \[
   \begin{array}{|c|c|c|}
   \hline
   i & P_i & N_i & P_i/N_i \\
   \hline
   0 & \int_{-\pi}^{\pi} L_0(x) \sin x \, dx & \int_{-\pi}^{\pi} L_0(x)^2 \, dx \\
   1 & \int_{-\pi}^{\pi} L_1(x) \sin x \, dx & \int_{-\pi}^{\pi} L_1(x)^2 \, dx \\
   2 & \int_{-\pi}^{\pi} L_2(x) \sin x \, dx & \int_{-\pi}^{\pi} L_2(x)^2 \, dx \\
   3 & \int_{-\pi}^{\pi} L_3(x) \sin x \, dx & \int_{-\pi}^{\pi} L_3(x)^2 \, dx \\
   \hline
   \end{array}
   \]
   (c) Define a function \( p(x) = (P_0/N_0)L_0(x) + (P_1/N_1)L_1(x) + (P_2/N_2)L_2(x) + (P_3/N_3)L_3(x). \)
   (d) Graph \( \sin(x) \) and \( p(x) = \left(\frac{315}{2\pi^2} - \frac{15}{2\pi^2}\right)x + \left(-\frac{525}{2\pi^2} + \frac{350}{2\pi^2}\right)x^3. \)
   (e) Integrate \( \int_{-\pi}^{\pi} |p(x) - \sin(x)| \, dx \)
   (f) Compare this error to the error from the Taylor polynomial of the same degree.