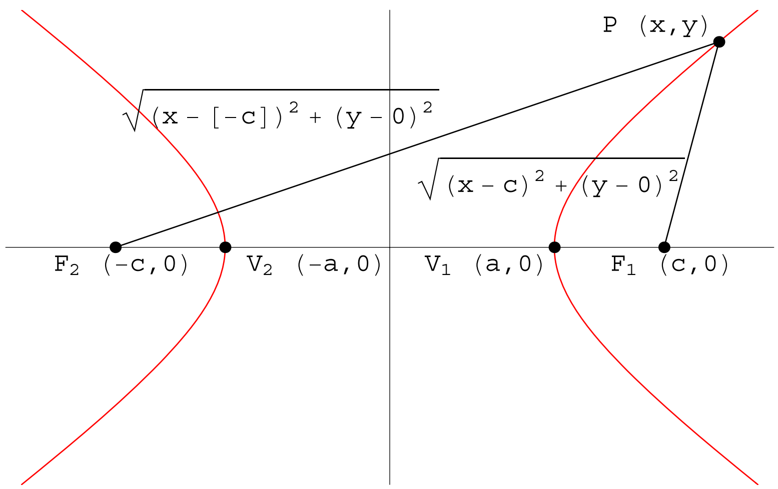
Discovering Hyperbolas

Definition

A set of points is a *hyperbola* if and only if
the difference between
the distance from any point P on the parabola to a fixed point F_1, called the focus and
the distance from that point P to a second focus F_2
is constant.

Diagram



Derivation

$$
\begin{aligned}
\sqrt{[x-(-c)]^2+y^2} - \sqrt{(x-c)^2+y^2} &= 2a. \\
\sqrt{(x+c)^2+y^2} &= 2a + \sqrt{(x-c)^2+y^2}. \\
(\sqrt{(x+c)^2+y^2})^2 &= (2a + \sqrt{(x-c)^2+y^2})^2. \\
(x+c)^2+y^2 &= 4a^2 + 4a\sqrt{(x-c)^2+y^2} + (x-c)^2+y^2 \\
x^2+2cx+c^2+y^2 &= 4a^2 + 4a\sqrt{(x-c)^2+y^2} + x^2-2cx+c^2+y^2. \\
4cx &= 4a^2 + 4a\sqrt{(x-c)^2+y^2}. \\
cx &= a^2 + a\sqrt{(x-c)^2+y^2}. \\
cx - a^2 &= +a\sqrt{(x-c)^2+y^2}. \\
(cx-a^2) &= (a\sqrt{(x-c)^2+y^2})^2. \\
c^2x^2 - 2a^2cx + a^4 &= a^2[(x-c)^2+y^2]. \\
c^2x^2 - 2a^2cx + a^4 &= a^2(x^2-2cx+c^2+y^2). \\
c^2x^2 - 2a^2cx + a^4 &= a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2. \\
c^2x^2 + a^4 &= a^2x^2 + a^2c^2 + a^2y^2. \\
a^4 - a^2c^2 &= a^2x^2 - c^2x^2 + a^2y^2. \\
(c^2-a^2)x^2 - a^2y^2 &= a^2(c^2-a^2). \\
b^2x^2 - a^2y^2 &= a^2b^2. \\
\frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1.
\end{aligned}
$$