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Deriving Change of Bases

$$\mathcal{B} = \{x^2, x, 1\}$$
$$\mathcal{C} = \{x^2+1, x^2+x, x+1\}$$

Note first the following coordinates.

$$\begin{aligned}
\left[x^2+x+5\right]_{\mathcal{B}} &= [1,1,5].\\
\left[x^2\right]_{\mathcal{C}} &= [1/2, 1/2, -1/2].\\
[x]_{\mathcal{C}} &= [-1/2, 1/2, 1/2].\\
[1]_{\mathcal{C}} &= [1/2, -1/2, 1/2].
\end{aligned}$$

$$\begin{aligned}
x^2+x+5 &= 1(x^2)+1(x)+5(1) && \text{expanded as linear combination of } \mathcal{B}\\
&= 1(1/2(x^2+1)+1/2(x^2+x)-1/2(x+1))+ \\
&\quad 1(-1/2(x^2+1)+1/2(x^2+x)+1/2(x+1))+ \\
&\quad 5(1/2(x^2+1)-1/2(x^2+x)+1/2(x+1)) && \vec{b}_i \text{ written as linear combinations of } \mathcal{C}\\
&= [1(1/2)+1(-1/2)+5(1/2)](x^2+1)+ \\
&\quad [1(1/2)+1(1/2)+5(-1/2)](x^2+x)+ \\
&\quad [1(-1/2)+1/(1/2)+5(1/2)](x+1) && \text{collecting in terms of } \vec{c}_i\\
&= 5/2(x^2+1)-3/2(x^2+x)+5/2(x+1).
\end{aligned}$$

Compare the arithmetic above (last two steps) to the calculation below.

$$\begin{aligned}
\begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} &= \begin{bmatrix} 1(1/2)+1(-1/2)+5(1/2) \\ 1(1/2)+1(1/2)+5(-1/2) \\ 1(-1/2)+1(1/2)+5(1/2) \end{bmatrix}\\
&= \begin{bmatrix} 5/2 \\ -3/2 \\ 5/2 \end{bmatrix}.
\end{aligned}$$